SOLUTIONS TO MIDTERM 2 PRACTICE PROBLEMS

(1) (a) To find $\mathcal{N}(A)$, we solve the matrix equation $Ax = 0$. Row reducing the matrix $[A\,|\,0\,]$ gives us $\begin{bmatrix} 1 & 2 & 1 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \end{bmatrix}$, so

$$\mathcal{N}(A) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \;:\; \begin{matrix} x_1 = -2x_2 - x_3 + 5x_5 \\ x_4 = -4x_5 \end{matrix} \right\}$$

$$= \left\{ x = \begin{bmatrix} -2x_2 - x_3 + 5x_5 \\ x_2 \\ x_3 \\ -4x_5 \\ x_5 \end{bmatrix} \right\} = \left\{ x = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 5 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

Thus the set $S = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$ is a spanning set for $\mathcal{N}(A)$, and since it is also linearly independent, it is a basis for $\mathcal{N}(A)$. (We can see this last fact since the matrix $\begin{bmatrix} -2 & -1 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix}$ row reduces easily to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.)

(b) From above, we can see that the recuded echelon form of A is the matrix $\begin{bmatrix} 1 & 2 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$. Since the leading 1's appear in the first and fourth columns, we know that the set $\{A_1, A_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\}$ forms a basis for $\mathcal{R}(A)$.

(c) The non-zero rows of the reduced echelon form of A form a basis for the row space of A, so $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \right\}$ is a basis.

(d) True. $\mathcal{N}(A) = \{\text{solutions to } \; Ax = 0\}$ and $\mathcal{N}(B) = \{\text{solutions to } \; Bx = 0\}$. These two matrix equations will have the same solutions if and only if their augmented matrices are row equivalent. Since A and B are row equivalent, so are $[A \mid 0]$ and $[B \mid 0]$. Thus {solutions to $Ax = 0$} = {solutions to $Bx = 0$}, and so A and B have the same null space.

(e) True. Row equivalent matrices do not in general have the same column space. However, from part (b), we can see that $\mathcal{R}(A) = \operatorname{Span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\7\end{bmatrix}\right\} = \mathbb{R}^2$ since the dimension of $\mathbb{R}^2$ is 2 and the set $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\7\end{bmatrix}\right\}$ is linearly independent (which means that since it has 2 elements, it is also a spanning set for $\mathbb{R}^2$ by Theorem 9 from section 3.5). We know that row equivalant matrices have the same rank, so any matrix B that's row equivalent to A will also have rank 2. Thus a basis for $\mathcal{R}(B)$ will also have 2 elements, and by the same argument as above, the span of that basis will be all of $\mathbb{R}^2$, so $\mathcal{R}(B) = \mathbb{R}^2$ as well.

(2) (a) The set $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

(b) The set $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ spans $\mathbb{R}^3$, but is not a basis, since it is not linearly independent.

(c) The set $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}$ is a linearly independent set in $\mathbb{R}^4$, but it is not a basis since it does not span $\mathbb{R}^4$. (For example, $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ is not in the span.)

(3) (a) Let $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, and $v_3 = \begin{bmatrix}0\\-2\\2\end{bmatrix}$. We only know what T does to v_1, v_2 and v_3, so to find what T does to $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ we first need to find a way to express e_1 and e_2 in terms of v_1, v_2 and v_3. That is, we need to solve $e_1 = a_1v_1 + a_2v_2 + a_3v_3$ for a_1, a_2 and a_3. This is equivalent to solving the matrix equation $Ax = e_1$, where $A = [\, v_1\ v_2\ v_3 \,]$. The augmented matrix is $\begin{bmatrix}1 & 0 & 0 & 1\\2 & 0 & -2 & 0\\3 & 1 & 2 & 0\end{bmatrix}$ which row reduces to $\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & -5\\0 & 0 & 1 & 1\end{bmatrix}$, which tells us that $e_1 = v_1 - 5v_2 + v_3$. Thus,

$$T(e_1) = T(v_1 - 5v_2 + v_3) = T(v_1) - 5T(v_2) + T(v_3)$$

$$= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Similarly, if we solve $e_2 = b_1 v_1 + b_2 v_2 + b_3 v_3$ we get the augmented matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix}$ which row reduces to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$ so $e_2 = v_2 - (1/2)v_3$ and

$$T(e_2) = T(v_2) - (1/2)T(v_3) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - (1/2) \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(b) The matrix A such that $T(x) = Ax$ for all x is the matrix $A = [T(e_1)\ T(e_2)\ T(e_3)]$, so from the information given and the calculations above, $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(4) (a) From the given information, we know that the vectors $w_1 = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ are both in the range of C. We can see that the set $\{w_1, w_2\}$ is linearly independent, and so the rank of C must be at least 2. Since rank$(C) \leq n = 3$, this means that rank$(C) = 2$ or 3.

Similarly, we can see that $u_1 = \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix}$ is in the null space of C, so the nullity of C is at least 1. Since rank$(C)+$ nullity$(C) = n = 3$, we must have that rank$(C) = 2$. Thus $\{w_1, w_2\}$ forms a basis for the range of C, i.e. $\mathcal{R}(A) = \text{Span}\{w_1, w_2\}$.

From this description of $\mathcal{R}(A)$, we can see that a vector $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in $\mathcal{R}(A)$ if an only if the equation $b = c_1 w_1 + c_2 w_2$ is consistent. The Augmented matrix for this system is $[\ w_1\ \ w_2 \mid b\ ] = \begin{bmatrix} 1 & 2 & b_1 \\ 4 & 1 & b_2 \\ 0 & 5 & b_3 \end{bmatrix}$ which row reduces to $\begin{bmatrix} 1 & 2 & b_1 \\ 0 & 1 & b_3/5 \\ 0 & 0 & b_2 - 4b_1 + (7/5)b_3 \end{bmatrix}$. Thus $b \in \mathcal{R}(A)$ if and only if $b_2 - 4b_1 + (7/5)b_3 = 0$.

So an algebraic specification for $\mathcal{R}(A)$ is

$$\mathcal{R}(A) = \left\{ b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3 \ :\ b_2 - 4b_1 + (7/5)b_3 = 0 \right\}$$

(b) From above, since rank$(C) = 2$, nullity$(C) = 1$, and so $\left\{ \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathcal{N}(A)$. Thus $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathcal{N}(A)$ if and only if $x = t \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix}$ for some $t \in \mathbb{R}$, and so

$$\mathcal{N}(A) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \ : \ \begin{matrix} x_1 = (9/2)x_3 \\ x_2 = (3/2)x_3 \end{matrix} \right\}$$

(c) The equation $Cx = v$ where $v = \begin{bmatrix} 8 \\ 11 \\ 15 \end{bmatrix}$ will have solutions if and only if $v \in \mathcal{R}(A)$. From above, we can use the algebraic specification for $\mathcal{R}(A)$ to check this: $11 - (4)(8) + (7/5)(15) = 11 - 32 + 21 = 0$, so v is in $\mathcal{R}(A)$ and so $Cx = v$ has at least 1 solution.

Let x_1 be one solution to $Cx = v$ (we know there's at least one.) Let $x_2 = x_1 + \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix}$. Then

$$Cx_2 = C(x_1 + \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix}) = Cx_1 + C \begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix} = Cx_1 + 0 = v$$

since $\begin{bmatrix} 9 \\ 3 \\ 2 \end{bmatrix} \in \mathcal{N}(A)$ and x_1 is a solution to $Cx = v$. This tells us that x_2 is also a solution to $Cx = v$. Thus since there are at least 2 solutions to $Cx = v$, there must be infinitely many.

(5) The data in the table gives us the over-constrained system

$$\begin{aligned} 4 &= (-2)^2a + (-2)b + c \\ 0 &= (-1)^2a + (-1)b + c \\ 2 &= (1)^2a + (1)b + c \\ 4 &= (2)^2a + (2)b + c \end{aligned}$$

which corresponds to the equation $Ax = b$ where $A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 4 \end{bmatrix}$. To find the best least-squares quadratic fit for the data, we need to find a least-squares solution to $Ax = b$. To do this, we solve the equation $A^TAx = A^Tb$. Now, $A^TA = \begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 4 \end{bmatrix}$ and $A^Tb = \begin{bmatrix} 34 \\ 2 \\ 10 \end{bmatrix}$, so the augmented matrix is $\begin{bmatrix} 34 & 0 & 10 & 34 \\ 0 & 10 & 0 & 2 \\ 10 & 0 & 4 & 10 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. This tells us that our least-squares solution to $Ax = b$ is $x^* = \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix} = \begin{bmatrix} 1 \\ 1/5 \\ 0 \end{bmatrix}$, so the best least-squares quadratic fit to the data is $y = t^2 + (1/5)t$.

(6) (a) False. There are many different bases for any subspace.
  (b) False. Any basis for W will have p elements, but W will usually have infinitely many elements.
  (c) False. For example, $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$, and $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a subspace of $\mathbb{R}^3$, but any basis for W will consist of one element which is a multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and this is not true for any of the vectors in B.
  (d) True. If $\dim(W) = n$, then W has a basis $B = \{w_1, \ldots, w_n\}$, and since $W \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$. Thus since B is a linearly independent set of n elements in $\mathbb{R}^n$, it is a basis for $\mathbb{R}^n$. Thus $\mathbb{R}^n = \text{Span}(B) = W$.
  (e) False. The subspace criterion says that the zero vector must be in W since W is a subspace. This means that the zero vector is not in V, so V fails the first condition of the subspace criterion and so is not a subspace.