Math 308 H
Midterm 1
January 30, 2008

Name: SOLUTIONS

1	25	
2	15	
3	20	
Total	60	

- You may use a scientific calculator during this examination. Other electronic devices are not allowed, and should be turned off for the duration of the exam.
- You may use one 8.5 by 11 inch page of notes. They must be handwritten.
- **Show all work for full credit.**
- You have 50 minutes to complete the exam.

1. Let C be a (4×4) matrix, and let $C \begin{bmatrix} -1 \\ 5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(a) How many solutions are there to the matrix equation $Cx = 0$? Explain your answer.

We are given that $\begin{bmatrix} -1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$ is a solution to $Cx = 0$, and we know that $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a solution, so since we have more than one solution, there must be infinitely many solutions.

(b) What can you say about the rank of C? Explain your answer.

Since we have infinitely many solutions, we must have that $n - r > 0$. Since $n = 4$, this means that $0 \leq r < 4$.

(c) Let c_1, c_2, c_3, and c_4 be the columns of C. Write c_1 as a linear combination of c_2, c_3, and c_4.

$C \begin{bmatrix} -1 \\ 5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ corresponds to the vector equation $-c_1 + 5c_2 - 2c_3 + c_4 = 0$, so $c_1 = 5c_2 - 2c_3 + c_4$.

(d) If B is a (3×4) matrix such that

$$Bc_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad Bc_3 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \quad Bc_4 = \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}$$

What is Bc_1?

$$Bc_1 = B(5c_2 - 2c_3 + c_4) = 5(Bc_2) - 2(Bc_3) + (Bc_4) = 5\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}.$$

(e) What is $(BC)\begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}$?

$$C\begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix} = 3c_1 - 2c_2 + c_3 + c_4, \text{ so } (BC)\begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix} = B\left(C\begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}\right) = B(3c_1 - 2c_2 + c_3 + c_4) =$$

$$3\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}.$$

2. Let A be a (5×4) matrix, and let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$. Suppose that we perform row operations on the augmented matrix $[\, A \mid b \,]$ to get the matrix

$$\begin{bmatrix} 1 & 2 & -3 & 1 & -b_1 + b_3 + b_5 \\ 0 & 1 & 4 & -2 & b_2 + b_3 + b_4 - b_5 \\ 0 & 0 & 1 & -1 & -3b_1 + 4b_2 + b_3 \\ 0 & 0 & 0 & 0 & 2b_1 - b_2 + b_4 \\ 0 & 0 & 0 & 0 & b_2 + 2b_3 - 3b_5 \end{bmatrix}$$

Note that this is NOT in reduced echelon form.

(a) Are the columns of the original matrix A linearly independent or linearly dependent? Explain how you can tell.

We can see that the rank of A is 3 (number of leading 1's). If the columns of the original matrix are $A_1, \ldots A_4$, then we want to solve the vector equation $x_1 A_1 + \cdots + x_4 A_4 = 0$ to see if there are non-trivial solutions. This vector equation is equivalent to the matrix equation $Ax = 0$, which is always consistent. Thus there will be $n - r = 4 - 3 = 1$ unconstrained variable in the solution, so there are non-trivial solutions. This means that the columns of A are linearly dependent.

(b) Only one of the following matrix equations represents a consistent system. Circle the consistent equation and explain how you can tell.

$$Ax = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \\ 5 \end{bmatrix} \qquad Ax = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 4 \\ -1 \end{bmatrix} \qquad Ax = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 0 \\ -2 \end{bmatrix}$$

For the system to be consistent, we need to make sure we don't have a row that looks like $[0\ 0\ 0\ 0\ c]$ where $c \neq 0$. That means we need to be sure that $2b_1 - b_2 + b_4 = 0$ AND that $b_2 + 2b_3 - 3b_5 = 0$. The only vector b that satisfies both these conditions is the one on the right.

(c) Solve the system that you circled in part (b). Give your answer in vector form.

Plugging in $b_1, \dots, b_5$ to the augmented matrix, we get $\begin{bmatrix} 1 & 2 & -3 & 1 & -7 \\ 0 & 1 & 4 & -2 & -4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Now, if we finish row reducing to get reduced echelon form, we have $\begin{bmatrix} 1 & 0 & 0 & -6 & 4 \\ 0 & 1 & 0 & 2 & -4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, which tells us that $x_1 = 4 + 6x_4$, $x_2 = -4 - 2x_4$, and $x_3 = 1 + x_4$. So our solutions look like $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ -2 \\ 1 \\ 1 \end{bmatrix}$ where x_4 can be anything.

3. Let B be a (3×3) matrix, and let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix}$.

(a) Let $D = [\, v_1\ v_2\ v_3 \,]$. Find D^{-1}.

If we row reduce the matrix $[D \mid I\,]$, we get $\begin{bmatrix} 1 & 0 & 0 & 13 & -3 & -3 \\ 0 & 1 & 0 & -6 & 1 & 2 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{bmatrix}$, so

$$D^{-1} = \begin{bmatrix} 13 & -3 & -3 \\ -6 & 1 & 2 \\ -4 & 1 & 1 \end{bmatrix}.$$

(b) Is the set $S = \{v_1, v_2, v_3\}$ linearly independent or linearly dependent? What about the set $T = \{v_1, v_2, v_3, v_4\}$ where v_4 is some other vector in $\mathbb{R}^3$? Explain your answers.

Since D is invertible, it's nonsingular, so it's columns are linearly independent. This means S is linearly independent. T is linearly dependent, since any set of 4 vectors in $\mathbb{R}^3$ is linearly dependent.

(c) Suppose that you know that

$$Bv_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \qquad Bv_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \qquad Bv_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Based on the information so far, can you determine if B is singular or non-singular? Explain your answer.

The matrix $[\ Bv_1\ \ Bv_2\ \ Bv_3\ ]$ is the matrix $B[\ v_1\ v_2\ v_3\ ] = BD$. We can seem almost immediately that BD is non-singluar (since $BD = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ easily row reduces to I). This means that both B and D are non-singluar.

(d) Find the matrix B.

$$B = (BD)D^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 13 & -3 & -3 \\ -6 & 1 & 2 \\ -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 26 & -6 & -6 \\ -18 & 3 & 6 \\ 4 & -1 & -1 \end{bmatrix}.$$