MATH 308 – Spring 2007 – Midterm 2

Name: ________________________________ Student ID Number: ______________

Directions:

- **You have 50 minutes to complete this exam.**
- This exam is worth 50 points with 5 possible extra credit points.
- Please leave all solutions in *EXACT* form; no decimal answers!
- Show all of your work and be as neat as possible.
- No calculators are allowed.

Problem #1: __________

Problem #2: __________

Problem #3: __________

Problem #4: __________

Problem #5: __________

Problem #6: __________

Total: __________

Good Luck!

(1) *(8+4 points)*

(a) Find the least-squares linear fit for the following data.

t	-1	0	1
$f(t)$	0	1	-1

Solution: If we attempt to fit a line to the above data we get the following inconsistent system:

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

So $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. This gives that

$$A^T A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } A^T \mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Hence, we need to solve the system of equations corresponding to the auugmented matrix,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}.$$

This gives that $c = 0$ and $m = -\frac{1}{2}$. The line is $y = -\frac{1}{2}t$.

(b) Is it possible to find a polynomial which passes through each of the above data points? Justify your answer.

Solution: Yes. The quadratic-fit augmented matrix is,

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

This system is consistent so there is a quadratic passing through the data points.

(2) *(12 points)* Let $W = Span(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4})$ where,

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and, } \mathbf{v_4} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

Find an orthogonal basis for W.

Solution: First we find a basis for W. Make the vectors the rows of a matrix and find the RREF:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, $\mathbf{w_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{w_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{w_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ is a basis for W. We apply Gram-Schmidt to get an orthogonal basis. Set $\mathbf{u_1} = \mathbf{w_1}$. Now,

$$\mathbf{u_2} = \mathbf{w_2} - a\mathbf{u_1}\,,$$

so, $a = -1$ which gives, $\mathbf{u_2} = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^T$. Next,

$$\mathbf{u_3} = \mathbf{w_3} - b\mathbf{u_1} - c\mathbf{u_2}\,,$$

so, $b = \frac{1}{2}$, $c = -\frac{1}{3}$ which gives, $\mathbf{u_3} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & 1 & -\frac{1}{6} \end{bmatrix}^T$. We have then that,

$$\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ 1 \\ -\frac{1}{6} \end{bmatrix}\}$$

is an orthogonal basis of W.

(3) *(2+2+2+2+2 points)* For the following circle True or False.

(a) Let A be an $m \times n$ matrix. If A has orthogonal columns then $n \leq m$.

False

(b) There exists a 7×7 matrix A such that the range of A equals the null-space of A.

False

(c) If A is an $n \times n$ matrix whose null-space is of dimension zero then A is nonsingular.

True

(d) If A is a 7×10 matrix then the maximum possible dimension of the range is 10.

False

(e) Every subspace has an orthonormal basis.

False

(4) *(4 points)* Let A be a 2×2 matrix such that the columns of A form an orthonormal basis of $\mathbb{R}^2$. Show that $A^T A = I$.

Solution: Write $A = [A_1\ A_2]$ where A_1 and A_2 are the columns of A. Then,

$$A^T A = \begin{bmatrix} A_1^T A_1 & A_1^T A_2 \\ A_2^T A_1 & A_2^T A_2 \end{bmatrix}.$$

But since $\{A_1, A_2\}$ is orthonormal, $A_1^T A_1 = 1 = A_2^T A_2$ and $A_1^T A_2 = 0 = A_2^T A_1$, so the result follows.

(5) *(6+6 points)* For each of the following matrices find an orthonormal basis for the null-space.

(a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution: A is already in RREF. We have, $x_1 = -x_2 - 2x_3$. In vector form this is,

$$\mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

So $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null-space of A. We apply Gram-Schmidt to find an orthogonal basis. Set $\mathbf{u_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{u_2} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - a \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. This gives that $a = 1$ and hence, $\mathbf{u_2} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. So, $\{\mathbf{u_1}, \mathbf{u_2}\}$ is an orthogonal basis for the null-space. To get an orthonormal basis we divide each vector by its length:

$$\{\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\},$$

is an orthonormal basis for the null-space.

(b) $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Solution: In this case we have, $x_1 = -2x_3$ and $x_2 = -x_3$. In vector form this is,

$$\mathbf{x} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix},$$

So $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ is a basis for the null-space of B. To get an orthonormal basis we need to divide by the length:

$$\{\frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}\},$$

is an orthonormal basis for the null-space of B.

(6) EXTRA CREDIT *(3+2 points)* Set $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(a) Find all vectors $\mathbf{v}$ such that $A\mathbf{v} = 2\mathbf{v}$.

Solution: We have that,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ v_2 + v_3 \\ 2v_3 \\ 2v_4 \end{bmatrix}.$$

Thus we need that $v_1 + v_2 = 2v_1$ and $v_2 + v_3 = 2v_3$. This gives that, $v_1 = v_2 = v_3$. So, $A\mathbf{v} = 2\mathbf{v}$ when,

$$\mathbf{v} = v_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) Show that $W = \{ \mathbf{v} \mid A\mathbf{v} = 2\mathbf{v} \}$ is a subspace of $\mathbb{R}^4$ and find its dimension.

Solution: Note that, $N(A - 2I) = W$. This implies W is a subspace. From above we see that the dimension of W is 2.