Homework 5 Supplemental Problem Solutions

1. The matrix $S = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$ is a reflection matrix. When a vector in $\mathbb{R}^3$ is multiplied by this matrix, the effect is to reflect the vector in a certain plane. Find an equation for that plane.

   (Hint: First try to find a normal vector to the plane. This is a vector $v \in \mathbb{R}$ that gets flipped by reflection, i.e. $Sv = -v$. To find such a v, we follow a process similar to that of Supplemental Problem 2 from HW2, i.e. $Sv = -v \Leftrightarrow Sv + v = 0 \Leftrightarrow (S+I)v = 0$.)

   To find a vector v such that $Sv = -v$, we solve the matrix equation $(S+I)x = 0$. The augmented matrix is

$$[S + I \mid 0] = \begin{bmatrix} 4/3 & 2/3 & 2/3 & 0 \\ 2/3 & 4/3 & -2/3 & 0 \\ 2/3 & -2/3 & 4/3 & 0 \end{bmatrix}$$

   which row reduces to

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   So we get $x_1 = -x_3$ and $x_2 = x_3$, and our solutions are of the form

$$\left\{ x = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} \right\} = \left\{ x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

   We're interested in any non-zero solution, so setting $x_3 = 1$ we get $v = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

   The plane that has $v = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ as a normal vector is all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

such that $v \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ which gives the condition $-x + y + z = 0$. This is the equation for our plane.

We can do a quick check to make sure this is correct. If we have a vector that lives in this plane, then reflecting across the plane should do nothing to that vector. That is, if w is a vector in this plane, then $Sw = w$, since Sw is the vector w reflected across the plane. The vector $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ lies in this plane, and

$$Sw = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6/3 \\ 3/3 \\ 3/3 \end{bmatrix} = w$$

So this is probably the right plane.

2. Sample Exam Question:

   Let $A = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 2 & 0 & 4 & 1 & 1 \\ 3 & 0 & 6 & 1 & 2 \end{bmatrix}$.

   Consider the following vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix}, \begin{bmatrix} 6 \\ 11 \\ 16 \end{bmatrix}$$

   (a) Determine which of the above vectors are in $\mathcal{N}(A)$.
   A is a (3×5) matrix, so $\mathcal{N}(A) \subseteq \mathbb{R}^5$, so we only need to check the 5-dimensional vectors. To check if a v vector is in $\mathcal{N}(A)$, we

need only check that $Av = 0$. So, since $A \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and

$A \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, those two vectors are in $\mathcal{N}(A)$. However,

$A \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 16 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ is not in $\mathcal{N}(A)$.

(b) Determine which of the above vectors are in $\mathcal{R}(A)$.
A is a (3×5) matrix, so $\mathcal{R}(A) \subseteq \mathbb{R}^3$, so we only need to check the 3-dimensional vectors. We know that $\mathcal{R}(A)$ is a subspace, so it always contains the 0 vector, so $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in $\mathcal{R}(A)$. Also, we know that $\mathcal{R}(A)$ is equal to the column space, so any vector that we can identify as a linear combination of the columns of A is in $\mathcal{R}(A)$. So $\begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} = 5A_1$ is in $\mathcal{R}(A)$, and $\begin{bmatrix} 6 \\ 11 \\ 16 \end{bmatrix} = 5A_1 + A_4$ is in $\mathcal{R}(A)$.

To see if $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is in $\mathcal{R}(A)$, we check to see if the matrix equation $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is consistent. The augmented matrix $\left[A \,\middle|\, \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \right]$ row reduces to

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The last row tells us that this system is inconsistent, and so $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is not in $\mathcal{R}(A)$.

3. Sample Exam Question:
Describe the null space of the (2×6) zero matrix as precisely as possible.

We let $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, the (2×6) zero matrix. To find $\mathcal{N}(A)$, we solve $Ax = 0$. This gives us the agumented matrix $[A \mid 0] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which gives us the equations $0 = 0$. Thus there are no restrictions on $x_1, \ldots, x_6$, and so every vector in $\mathbb{R}^6$ is in the null space of A. Thus $\mathcal{N}(A) = \mathbb{R}^6$.