

Homework 6 Supplemental Problem Solutions

1. Sample Exam Question:

In this question, A is a certain $(m \times n)$ matrix and $u_1, u_2, u_3, w_1, w_2, w_3,$ and w_4 are certain vectors. Suppose you are told that $\{u_1, u_2, u_3\}$ is a basis for $\mathcal{R}(A)$ and that $\{w_1, w_2, w_3, w_4\}$ is a basis for $\mathcal{N}(A)$. You are told nothing else about the matrix A or the vectors $u_1, u_2, u_3, w_1, w_2, w_3, w_4$. On the basis of this information, either determine the following quantities precisely or find an inequality that the quantity satisfies. Explain your answers briefly.

(a) $\text{rank}(A)$

(b) n

(c) m

(d) The number of entries in the vector w_3 .

(e) The number of solutions to the matrix equation $Ax = u_2$.

(a) We know that $\text{rank}(A) = \dim(\mathcal{R}(A))$, and since $\{u_1, u_2, u_3\}$ is a basis for $\mathcal{R}(A)$, $\text{rank}(A) = 3$.

(b) The nullity of A is $\text{nullity}(A) = \dim(\mathcal{N}(A))$, so since $\{w_1, w_2, w_3, w_4\}$ is a basis for $\mathcal{N}(A)$, $\text{nullity}(A) = 4$. Now, by the rank-nullity theorem, $\text{rank}(A) + \text{nullity}(A) = n$, so $n = 7$.

(c) We know that $m \geq \text{rank}(A)$, since $\text{rank}(A) =$ the number of non-zero rows in the REF of A , and $m =$ the number of row of A (and the REF form of A). So $m \geq 3$. This is all we can say about m .

(d) the vector $w_3 \in \mathcal{N}(A)$, and we know that since A is an $(m \times n)$ matrix that $\mathcal{N}(A) \subseteq \mathbb{R}^n$. Thus $w_3 \in \mathbb{R}^n$, and so w_3 has 7 entries (from part (b)).

(e) Since $u_2 \in \mathcal{R}(A)$, the system $Ax = u_2$ is consistent. Thus the number of solutions to $Ax = u_2$ is ≥ 1 . Suppose that v is one such solution (there's at least 1), so $Av = u_2$. Note that $A(v + w_1) = Av + Aw_1 = u_2 + 0 = u_2$ since w_1 is in the null space of A . Thus $v + w_1$ is also a solution to $Ax = u_2$. Since we know there are two solutions, there must be infinitely many solutions.

2. Sample Exam Question:

$$\text{Let } A = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}.$$

Is it true that every vector in the range of A is also in the null space of A ?

First we find the range of A . Let $v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. We want to determine when the system $Ax = v$ is consistent. The augmented matrix is $[A | v] = \begin{bmatrix} 3 & -1 & a \\ 9 & -3 & b \end{bmatrix}$ and row reduces to $\begin{bmatrix} 1 & -1/3 & a/3 \\ 0 & 0 & b - 3a \end{bmatrix}$. Thus $v \in \mathcal{R}(A)$ if and only if $b - 3a = 0$, i.e. $b = 3a$. So $\mathcal{R}(A) = \left\{ v \in \mathbb{R}^2 \mid v = \begin{bmatrix} a \\ 3a \end{bmatrix} \text{ } a \in \mathbb{R} \right\}$.

Now we check to see if the vectors in $\mathcal{R}(A)$ are also in $\mathcal{N}(A)$. An arbitrary vector in $\mathcal{R}(A)$ looks like $v = \begin{bmatrix} a \\ 3a \end{bmatrix}$ for some $a \in \mathbb{R}$.

$$Av = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} a \\ 3a \end{bmatrix} = \begin{bmatrix} 3a - 3a \\ 9a - 9a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

no matter what a is. Thus every vector in $\mathcal{R}(A)$ is also in $\mathcal{N}(A)$.

3. Sample Exam Question:

Answer true or false. Justify your answers by providing a counter example if the statement is false or explain why the statement is true.

- (a) If $S = \{x_1, \dots, x_k\}$ is a subset of \mathbb{R}^n and $k \leq n$, then S is a linearly independent set.
- (b) If $S = \{x_1, \dots, x_k\}$ is a subset of \mathbb{R}^n and $k > n$, then S is a linearly dependent set.
- (c) If $S = \{x_1, \dots, x_k\}$ is a subset of \mathbb{R}^n and $k < n$, then S is not a spanning set for \mathbb{R}^n .
- (d) If $S = \{x_1, \dots, x_k\}$ is a subset of \mathbb{R}^n and $k \geq n$, then S is a spanning set for \mathbb{R}^n .
- (a) FALSE. Any set with the zero vector is linearly dependent, so $S = \{0\} \subseteq \mathbb{R}^2$ is a counter example.

- (b) TRUE. The dimension of \mathbb{R}^n is n , so Theorem 9 from Section 3.5 tells us that any set with more than n vectors in \mathbb{R}^n must be linearly dependent.
- (c) TRUE. Again, by Theorem 9 from Section 3.5 we know that any set with less than n vectors in \mathbb{R}^n is not a spanning set.
- (d) FALSE. Consider the set $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$. This set has 4 elements, but $\text{Sp}(S) = \text{Sp} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, which is not all of \mathbb{R}^3 , so S is not a spanning set for \mathbb{R}^3 .