Problem 1. (a) $\nabla f = (\cos \pi y, -\pi x \sin \pi y + e^z, ye^z)$, so $\nabla f(2,3,1) = -\mathbf{i} + e\mathbf{j} + 3e\mathbf{k}$.

(b) Let $\mathbf{w}$ be the vector from (2,3,1) to (5,3,5). Then $\mathbf{r}'(0) = 2\dfrac{\mathbf{w}}{|\mathbf{w}|} = 2\dfrac{(3,0,4)}{\sqrt{9+0+16}} = (\frac{6}{5}, 0, \frac{8}{5})$,

$$\text{so} \qquad \frac{d}{dt} f(\mathbf{r}(t)) = \nabla f \cdot \mathbf{r}'(0) = (-1, e, 3e) \cdot (\frac{6}{5}, 0, \frac{8}{5}) = \frac{24e-6}{5}.$$

Problem 2. Let C_1 and C_2 be the two line segments.
Parametrize: C_1, $\mathbf{r}(t) = (t,t,t)$ for $0 \le t \le 1$, and C_2, $\mathbf{r}(t) = (1, 1-t, 1)$ for $0 \le t \le 1$.

$$\text{mass } = \int_C (2-z)ds = \int_{C_1} (2-t)\sqrt{1+1+1}dt + \int_{C_2} (2-1)\sqrt{0+1+0}dt = \ldots = \sqrt{3}\frac{3}{2} + 1$$

Problem 3. (a) The components $P = x^3 - 2xy^3$ and $Q = -3x^2y^2$ are defined and continuously differentiable everywhere on the xy-plane, which is simply connected (has "no holes"). So $\mathbf{F}$ will be conservative if $\partial P/\partial y = \partial Q/\partial x$. Both these partials are $-6xy^2$, so $\mathbf{F}$ is conservative.

(b) If f is a potential for $\mathbf{F}$, then $\dfrac{\partial f}{\partial x} = P = x^3 - 2xy^3$. Integrating with respect to x, $f(x,y) = \frac{x^4}{4} - x^2y^3 + g(y)$. Then computing $\dfrac{\partial f}{\partial y}$ from this and setting it equal to $Q = -3x^2y^2$, we get $\frac{\partial f}{\partial y} = -3x^2y^2 + g'(y) = -3x^2y^2$. Thus $g(y)$ is constant, and $f(x,y) = \frac{x^4}{4} - x^2y^3$ is a potential for $\mathbf{F}$. (One potential function suffices as an answer, but more generally, $f(x,y) = \frac{x^4}{4} - x^2y^3 + k$ for any constant k is also a potential for $\mathbf{F}$.)

(c) Let C be the given parametrized curve, $\mathbf{r} = (\cos^3 t, \sin^3 t)$. Because we know a potential for $\mathbf{F}$ from (b), we can use the Fundamental Theorem for Line Integrals and compute

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(\pi/2)) - f(\mathbf{r}(0)) = f(0,1) - f(1,0) = (0-0) - (\frac{1}{4} - 0) = -\frac{1}{4}.$$

Problem 4. (a) Idea: The equation for the plane, $y + z = 5$, will give me the formula for z if I have one for y (or vice versa). So I should pick formulas for x and y that make sense for the cylinder, going completely around it; for instance, $x(t) = 3\cos t$ and $y(t) = 3\sin t$. Thus $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 5 - 3\sin t\rangle$ for $0 \le t \le 2\pi$ will work.

(b) $\mathbf{r}'(t) = \langle -3\sin t, 3\cos t, -3\cos t\rangle$

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \langle 3\cos t, 6\sin t, -4\rangle \cdot \langle -3\sin t, 3\cos t, -3\cos t\rangle dt \\
&= \int_0^{2\pi} (-9\cos t \sin t + 18 \cos t \sin t + 12\cos t)dt \\
&= \int_0^{2\pi} (9\cos t \sin t + 12\cos t)dt = \frac{9}{2}\sin^2 t - 12\sin t\Big|_0^{2\pi} = 0
\end{aligned}$$

(If I hadn't said "Use your parametrization to compute," what other way might you have done this integral?)