Math 324B
FIRST PRACTICE EXAM SOLUTIONS

1. (a) The line and the parabola intersect where $3x+4=4-x^2$, i.e., $x=-3$ and $x=0$, where $y=-5$ and $y=4$ respectively. So

$$\iint_D 2x\,dA = \int_{-3}^{0}\int_{3x+4}^{4-x^2} 2x\,dy\,dx = \int_{-5}^{4}\int_{-\sqrt{4-y}}^{(y-4)/3} 2x\,dx\,dy.$$

The first way gives $\int_{-3}^{0} 2x(-x^2-3x)\,dx = [-\frac{1}{2}x^4-2x^3]_{-3}^{0} = -\frac{27}{2}$, and the second way gives $\int_{-5}^{4}[(\frac{1}{3}(y-4))^2-(4-y)]\,dy = [\frac{1}{27}(y-4)^3+\frac{1}{2}(y-4)^2]_{-5}^{4} = -\frac{27}{2}$.

2. For $dz\,dy\,dx$, the base of the solid is the triangle in the xy-plane bounded by the coordinate axes and the line $3x+2y=6$, so

$$\iiint_E f(x,y,z)\,dV = \int_0^2\int_0^{(6-3x)/2}\int_0^{6-3x-2y} f(x,y,z)\,dz\,dy\,dx.$$

For $dy\,dx\,dz$, the base of the solid in the xz-plane is the triangle bounded by the coordinate axes and the line $3x+z=6$, so

$$\iiint_E f(x,y,z)\,dV = \int_0^6\int_0^{(6-z)/3}\int_0^{(6-3x-z)/2} f(x,y,z)\,dy\,dx\,dz.$$

For $dx\,dz\,dy$, the base of the solid in the yz-plane is the triangle bounded by the coordinate axes and the line $2y+z=6$, so

$$\iiint_E f(x,y,z)\,dV = \int_0^3\int_0^{6-2y}\int_0^{(6-2y-z)/3} f(x,y,z)\,dx\,dz\,dy.$$

3. The line $y=x/\sqrt{3}$ in the first quadrant is given by $\theta=\pi/6$ in polar coordinates, and the positive y-axis is given by $\theta=\pi/2$, so the integral is

$$\int_{\pi/6}^{\pi/2}\int_0^1 e^{-r^2}r\,dr\,d\theta = \frac{\pi}{3}[-\tfrac{1}{2}e^{-r^2}]_0^1 = \frac{\pi}{6}(1-e^{-1}).$$

4. In cylindrical coordinates the paraboloid is $z=r^2$, so

$$m = \int_0^{2\pi}\int_0^2\int_{r^2}^4 (8-2z)r\,dz\,dr\,d\theta = 2\pi\int_0^2 [8z-z^2]_{r^2}^4 r\,dr = 2\pi\int_0^2 (r^5-8r^3+16r)\,dr = \frac{64\pi}{3}.$$

5. (a) The limits of integration are $1 \le \rho \le 2$, $0 \le \phi \le \frac{1}{2}\pi$, $0 \le \theta \le \frac{1}{2}\pi$. (b) The mass is $\frac{1}{8}$ of the volume of the outer sphere minus the volume of the inner sphere, i.e., $\frac{1}{8} \cdot \frac{4\pi}{3}(2^3 - 1^3) = \frac{7\pi}{6}$. Also, $\overline{x} = \overline{y} = \overline{z}$ because E is symmetric under any permutation of the coordinates (i.e., its description doesn't change if you interchange the labels x, y, and z). The easiest one to compute in spherical coordinates is $\overline{z}$:

$$\overline{z} = \frac{1}{m} \iiint z\, dV = \frac{6}{7\pi} \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta = \frac{6}{7\pi} [\theta]_0^{\pi/2} [\tfrac{1}{2} \sin^2 \phi]_0^{\pi/2} [\tfrac{1}{4}\rho^4]_1^2$$

which equals $\frac{45}{56}$, so the center of mass is $(\frac{45}{56}, \frac{45}{56}, \frac{45}{56})$.

6. (a) Clearly $x + y = u$, so $v = y/u = y/(x+y)$. The boundary of R consists of pieces of the lines $x + y = 1$, $x + y = 3$, $y = 0$, and $x = 0$. These correspond to the lines $u = 1$, $u = 3$, $v = 0$, and $v = 1$, so R corresponds to the rectangle $1 \le u \le 3$, $0 \le v \le 1$. (b) In terms of u and v the integrand $1/(x+y)^2$ is $1/u^2$, and

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} 1-v & -u \\ v & u \end{pmatrix} = (1-v)u + vu = u,$$

so the integral becomes $\int_0^1 \int_1^3 (1/u^2) u \, du \, dv = \int_0^1 \int_1^3 (1/u) du \, dv = \ln 3$.