Math 324B
FINAL PRACTICE EXAM SOLUTIONS

1. In spherical coordinates the sphere is $\rho = \sqrt{2}$ and the cone is $\rho\cos\phi = \rho\sin\phi$, i.e., $\cos\phi = \sin\phi$, i.e., $\phi = \frac{1}{4}\pi$. Also $x = \rho\sin\phi\cos\theta$, $z = \rho\cos\phi$, and $dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$, so

$$\iiint_E e^{xz}\, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} e^{\rho^2\cos\phi\sin\phi\sin\theta}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$$

2. The iterated integral represents the double integral over the region D between the line $y = 2x$ and the parabola $y = 3 - x^2$, which meet at $(-3, -6)$ and $(1, 2)$. So as an iterated integral in the opposite order, it is $\int_{-6}^{2}\int_{-\sqrt{3-y}}^{y/2} f(x,y)\, dx\, dy + \int_2^3\int_{-\sqrt{3-y}}^{\sqrt{3-y}} f(x,y)\, dx\, dy$. (This should be clear if you draw a sketch of D.)

3. The sides of the triangle are the lines $y = 0$, $y = -x$, and $y = x + 2$. The inverse transformation of $u = x + y$, $v = x - y$ is $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$, so the three lines just described correspond to the lines $u = v$, $u = 0$, and $v = -2$, and the image of D is the triangle with vertices $(0,0)$, $(0,-2)$, and $(-2,-2)$. Also, the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\tfrac{1}{2},$$

and its absolute value (which is what you need) is $\frac{1}{2}$. Thus

$$\iint_D \cos\frac{\pi(x+y)}{2(x-y)}\, dA = \int_{-2}^{0}\int_v^0 \cos\frac{\pi u}{2v}\,\tfrac{1}{2}\, du\, dv = \int_{-2}^{0}\frac{v}{\pi}\sin\frac{\pi u}{2v}\Big|_v^0 dv = -\frac{1}{\pi}\int_{-2}^{0} v\, dv = \frac{2}{\pi}.$$

4. We have $\operatorname{div}\mathbf{F} = 2x + 0 + x = 3x$, so by the divergence theorem,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E 3x\, dV = \int_0^6\int_0^{(6-x)/2}\int_0^{(6-x-2y)/3} 3x\, dz\, dy\, dx \\ &= \int_0^6\int_0^{(6-x)/2}(6-x-2y)x\, dy\, dx = \int_0^6 \left[x(6-x)y - xy^2\right]_0^{(6-x)/2} dx \\ &= \int_0^6 \tfrac{1}{4}x(6-x)^2\, dx = \left[\tfrac{9}{2}x^2 - x^3 + \tfrac{1}{16}x^4\right]_0^6 = 162 - 216 + 81 = 27.\end{aligned}$$

5. (a) $\operatorname{curl}\mathbf{F} = \mathbf{0}$ and $\operatorname{div}\mathbf{F} = \frac{1}{2}e^{x/2} - 25x\cos(3y+4z) - 2z$.
(b) Yes, $f(x,y,z) = 2e^{x/2} + x\sin(3y+4z) - \frac{1}{3}z^3 + c$.
(c) No, because $\operatorname{div}\mathbf{F} \neq 0$.

6. For the integral over C_1, use Green's theorem. (The orientation is "wrong," so there's an extra minus sign.) Denoting the region inside the ellipse by D,

$$\int_{C_1} (xy\,dx - x^2\,dy) = -\iint_D \left[\frac{\partial(-x^2)}{\partial x} - \frac{\partial(xy)}{\partial y}\right] dA = \iint_D 3x\,dA = 0$$

because $3x$ is an odd function and D is symmetric about the line $x = 0$.

There are a couple of ways to do the integral over C_2. You can take y as the parameter (running backwards from 3 to -3); then $x = \frac{2}{3}\sqrt{9-y^2}$ and $dx = -\frac{2}{3}(y/\sqrt{9-y^2})\,dy$, so

$$\int_C (xy\,dx - x^2\,dy) = \int_3^{-3} \tfrac{4}{9}[-y^2 - (9-y^2)]\,dy = \tfrac{4}{9}(-9)(-6) = 24.$$

Or, you can you can use trig functions to parametrize, say $x = 2\sin t$, $y = 3\cos t$, $0 \le t \le \pi$ (other variations are possible). Then $dx = 2\cos t\,dt$ and $dy = -3\sin t\,dt$, so

$$\int_C (xy\,dx - x^2\,dy) = \int_0^\pi (12\cos^2 t\sin t + 12\sin^3 t)\,dt = \int_0^\pi 12\sin t\,dt = -12\cos t\Big|_0^\pi = 24.$$

For the scalar line integral, these two parametrizations give

$$\int_C x\,ds = \int_{-3}^{3} \frac{2}{3}\sqrt{9-y^2}\sqrt{\frac{4y^2}{9(9-y^2)}+1}\,dy = \int_0^\pi 3\sin t\sqrt{4\cos^2 t + 9\sin^2 t}\,dt.$$

(Yes, the second integral is $\int_{-3}^{3}$. We have $ds = \sqrt{dx^2+dy^2}$, which equals $\sqrt{(dx/dy)^2+1}\,dy$ only if the increment dy is positive, i.e., y goes from smaller to larger. Otherwise there's a minus sign that compensates for reversing the limits of integration.)

7. The surface is parametrized by $\mathbf{r}(\theta, z) = \sqrt{1+z^2}(\cos\theta)\mathbf{i} + \sqrt{1+z^2}(\sin\theta)\mathbf{j} + z\mathbf{k}$ ($0 \le \theta \le 2\pi$, $0 \le z \le 1$), so one calculates that $\mathbf{r}_\theta \times \mathbf{r}_z = \sqrt{1+z^2}(\cos\theta)\mathbf{i} + \sqrt{1+z^2}(\sin\theta)\mathbf{j} - z\mathbf{k} = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$ (with the right orientation: the horizontal part $x\mathbf{i} + y\mathbf{j}$ points outward). Thus $\mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z) = x^2 + y^2 - z^2$, which equals 1 on the surface S, so

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{2\pi} 1\,d\theta\,dz = 2\pi.$$

Also $dS = |\mathbf{r}_\theta \times \mathbf{r}_z|\,d\theta\,dz = \sqrt{x^2+y^2+z^2}\,d\theta\,dz = \sqrt{1+2z^2}\,d\theta\,dz$, so

$$\iint_S z\,dS = \int_0^1 \int_0^{2\pi} z\sqrt{1+2z^2}\,d\theta\,dz = 2\pi \cdot \tfrac{1}{4} \cdot \tfrac{2}{3}(1+2z^2)^{3/2}\Big|_0^1 = \frac{\pi}{3}(3^{3/2} - 1).$$

8. Use Stokes: a bit of calculation shows that $\operatorname{curl}\mathbf{F} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{r}_u \times \mathbf{r}_v = -6u\mathbf{i} + 2u\mathbf{j} + 2\mathbf{k}$ (the correct orientation), so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 (-2u + 6)\,du\,dv = 5.$$