1. (4 points) Suppose $f(x, y) = x^2y + y^2$, and $x = x(u, v)$ and $y = y(u, v)$ are functions of u and v, with

$$x(1,2) = 3 \qquad \frac{\partial x}{\partial u}(1,2) = -1 \qquad y(1,2) = 1 \qquad \frac{\partial y}{\partial u}(1,2) = 2,$$

Find $\dfrac{\partial f}{\partial u}$ when $u = 1$ and $v = 2$.

Solution:

$$\frac{\partial f}{\partial u}(1,2) = \frac{\partial f}{\partial x}(3,1)\frac{\partial x}{\partial u}(1,2) + \frac{\partial f}{\partial y}(3,1)\frac{\partial y}{\partial u}(1,2) = -6 + 22 = 16.$$

2. Let $g(x, y) = x \sin y$.

(a) (3 points) Determine the directional derivative $D_{\mathbf{u}}g(1,0)$ if $\mathbf{u} = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

Solution: $\nabla f = (\sin y, x \cos y) = (0,1)$ when $x = 1$ and $y = 0$. The direction derivative is $(0,1) \cdot \mathbf{u} = 2/\sqrt{5}$.

(b) (3 points) Find a unit vector $\mathbf{v}$ so that $D_{\mathbf{v}}g(1,0) < -\frac{3}{4}$.

Solution: The unit vector in the opposite direction of the gradient is $(0,-1)$ and makes the directional derivative as small as possible, so if anything is going to work, it will. Sure enough, $D_{(0,-1)}f(1,0) = -1$.

3. (10 points) Let E be the solid bounded by the following four planes:

$$x = 0 \qquad y = 0 \qquad z = 0 \qquad 2x + 2y + z = 4$$

Find the x-coordinate of the center of mass if the solid has constant density.

Solution: We calculate

$$\overline{x} = \frac{\left(\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} x\,dz\,dy\,dx\right)}{\left(\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} dz\,dy\,dx\right)} = \frac{4/3}{8/3} = \frac{1}{2}$$

4. Consider the region whose volume is naturally given by the integral

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} 1\,dz\,dy\,dx.$$

(a) (5 points) Write an integral in cylindrical coordinates that computes the volume of the same region. *Do not evaluate the integral.*

Solution: This is half the space inside the cylinder $x^2 + y^2 = 1$ and outside the cone $z^2 = x^2 + y^2$. In cylindrical coordinates, this is

$$\int_0^\pi \int_0^1 \int_0^r r\,dz\,dr\,d\theta.$$

(b) (5 points) Write an integral in spherical coordinates that computes the volume of the same region. *Do not evaluate the integral.*

Solution: The equation of the cylinder is $\rho = 1/\sin\phi$, so we have

$$\int_0^\pi \int_{\pi/4}^{\pi/2} \int_0^{1/\sin\phi} \rho^2 \sin\phi\,d\rho\,d\phi,\,d\theta.$$

5. (a) (8 points) Compute the integral

$$\iiint_R yz^2\,dV$$

where R is one of the four (you choose) regions bounded by the cylinder $x^2+y^2 = 1$ and the three planes $z = 2x$, $z = 0$, and $y = 0$.

Solution: This is best done using rectangular or cylindrical coordinates. I will do the piece in the first octant. We have

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{2x} yz^2\,dz\,dy\,dx = \int_0^{\pi/2} \int_0^1 \int_0^{2r\cos\theta} r^2 z^2 \sin\theta\,dz\,dr\,d\theta = \frac{1}{9}.$$

(b) (2 points) Now use symmetry to determine the value of the same integral over each of the 4 regions:

Solution: The regions are reflections of one another and the integrand is symmetric with respect to x and z. When y is negative, however, we will get the negative of our answer:

	$x \le 0, z \le 0$	$x \ge 0, z \ge 0$
$y \le 0$	$-1/9$	$-1/9$
$y \ge 0$	$1/9$	$1/9$

6. (10 points) Compute the integral

$$\int_0^1 \int_0^{1-x} \exp\left(\frac{x-y}{x+y}\right) dy\, dx$$

using the change of coordinates $u = x - y$, $v = x + y$.

Solution: Solving for x and y we get $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$, so the Jacobian is $\frac{1}{2}$. Before changing coordinates, the domain of integration was a triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Since the change of variables is linear, I know that the new domain of integration will also be a triangle, with vertices $(0,0)$, $(1,1)$, and $(-1,1)$. So we have

$$\int_0^1 \int_0^{1-x} \exp\left(\frac{x-y}{x+y}\right) dy\, dx = \frac{1}{2} \int_0^1 \int_{-v}^{v} e^{u/v}\, du\, dv = \frac{1}{4}\left(e - e^{-1}\right).$$