Problem 1 (20 points) Evaluate the following integrals.

(a) $I = \int_D \cos(x^2 + y^2)\, dA$, where $D = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 3\}$.

$$I = \int_0^{2\pi} \int_0^{\sqrt{3}} \cos(r^2)\, r\, dr\, d\theta$$

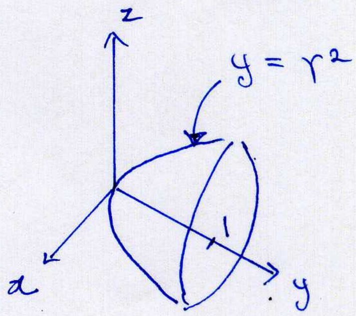
$$= 2\pi \int_0^{3} \cos(u) \frac{du}{2} \qquad \left(\begin{array}{c} u = r^2 \\ du = 2r\,dr \end{array} \right)$$

$$= \pi \sin(3).$$

(b) $I = \int_E x^2 + z^2\, dV$, where $E \subseteq \mathbb{R}^3$ is the solid bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 1$. (Hint: use cylindrical coordinates switching the roles of y and z.)

Use the coordinates

$$\begin{cases} x = r\cos(\theta) \\ z = r\sin(\theta) \\ y = y \end{cases}$$

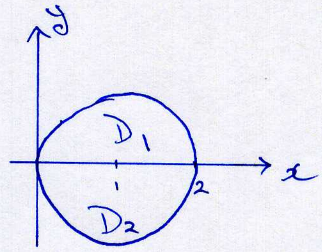


$$I = \int_0^{2\pi} \int_0^1 \int_{r^2}^{1} r^2 \cdot r\, dy\, dr\, d\theta$$

$$= 2\pi \int_0^1 r^3(1 - r^2)\, dr$$

$$= \frac{\pi}{6}$$

(c) $I = \int_D y\, dA$, where $D = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \leq 1\}$.

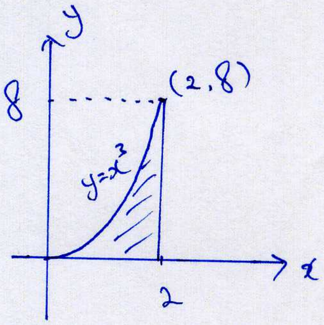


Since the integrand is an odd function in y,

$$\int_{D_1} y\, dA = -\int_{D_2} y\, dA$$

$$\Rightarrow I = 0.$$

(d) $I = \int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$.



Switch the order of integration:

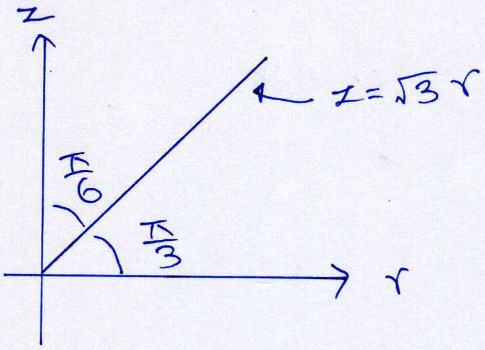
$$I = \int_0^2 \int_0^{x^3} e^{x^4}\, dy\, dx$$

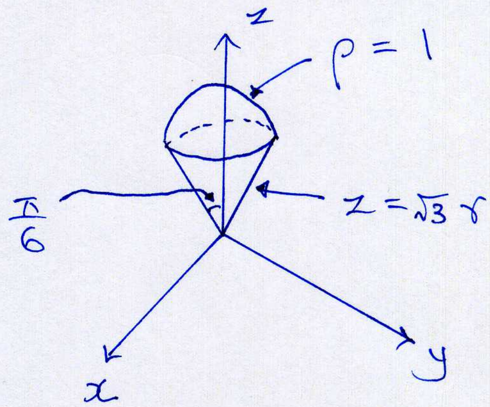
$$= \int_0^2 e^{x^4} \cdot x^3\, dx$$

$$= \int_0^{16} e^u \frac{du}{4} \quad \left(\begin{array}{l} u = x^4 \\ du = 4x^3 dx \end{array} \right)$$

$$= \frac{1}{4}(e^{16} - 1).$$

Problem 2 (10 points) Find the volume of the solid $E \subseteq \mathbb{R}^3$ bounded by the cone $z = \sqrt{3(x^2+y^2)}$ and the sphere $x^2+y^2+z^2=1$. (You are given that $\tan\frac{\pi}{3} = \sqrt{3}$.)





Use spherical coordinates:

$$\mathrm{vol}(E) = \int_E 1\, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^1 \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= 2\pi\left(\int_0^{\frac{\pi}{6}} \sin(\phi)\, d\phi\right)\left(\int_0^1 \rho^2\, d\rho\right)$$

$$= 2\pi\left(1-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3}\right)$$

$$= \frac{\pi}{3}\left(2-\sqrt{3}\right)$$ //

Problem 3 (10 points) The joint density function of two random variables X and Y is given by

$$f(x,y) = \begin{cases} C\, e^{-x} e^{-\frac{y}{2}} & x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant C.

$$\int_{\mathbb{R}^2} e^{-x} e^{-\frac{y}{2}}\, dA = \left(\int_0^\infty e^{-x}\, dx\right)\left(\int_0^\infty e^{-\frac{y}{2}}\, dy\right)$$

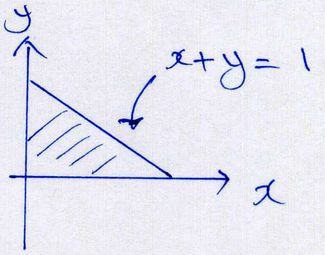
$$= \left[-e^{-x}\right]_{x=0}^{\infty} \cdot \left[-2e^{-\frac{y}{2}}\right]_{y=0}^{\infty}$$

$$= (1)(2)$$

$$= 2$$

$$\Rightarrow C = \frac{1}{2} \quad \left(\because \int_{\mathbb{R}^2} f(x,y)\, dA = 1\right)$$

(b) What is the probability of the event $X + Y \leq 1$?



$$P(X+Y \leq 1)$$

$$= \int_{\{x+y \leq 1\}} f(x,y)\, dA$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2} e^{-x} e^{-\frac{y}{2}}\, dy\, dx$$

$$= \int_0^1 -e^{-\frac{1}{2}-\frac{x}{2}} + e^{-x}\, dx$$

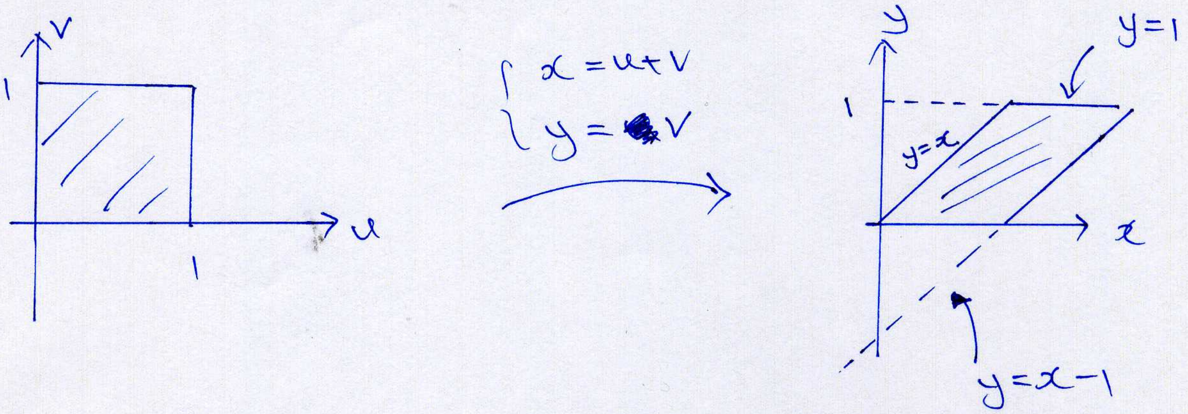
$$= e^{-1} - 2e^{-\frac{1}{2}} + 1$$

$$\left(\text{or} = \frac{(\sqrt{e}-1)^2}{e}\right)$$

Problem 4 (10 points) Use the change of variables

$$\begin{cases} x = u + v \\ y = v \end{cases}$$

to evaluate the double integral $I = \int_R (x-y)^{324} y \, dA$, where R is the parallelogram bounded by the lines $y = x$, $y = x - 1$, $y = 0$ and $y = 1$.



Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$$I = \int_0^1 \int_0^1 u^{324} v \cdot 1 \, du \, dv$$

$$= \left(\int_0^1 u^{324} du\right)\left(\int_0^1 v \, dv\right)$$

$$= \frac{1}{325} \cdot \frac{1}{2}$$

$$= \frac{1}{650}$$