Problem 1 (30 points) Evaluate the following integrals.

(a) $I = \int_C y\,dx - x^2 dy$, where C is the curve from $(0,0)$ to $(1,1)$ along the parabola $y = x^2$.

$\gamma(t) = (t, t^2), \quad 0 \le t \le 1$

$I = \int_0^1 t^2 - t^2(2t)\,dt$

$= \int_0^1 t^2 - 2t^3\,dt$

$= -\frac{1}{6}$ //

(b) $I = \int_S F \cdot dS$, where $F(x,y,z) = (1,1,1)$ and $S \subseteq \mathbb{R}^3$ is the surface parametrized by $\Gamma(u,v) = (u^2, v^2, u+v)$, $0 \le u \le 1$, $0 \le v \le 1$.

$\Gamma_u = (2u, 0, 1)$

$\Gamma_v = (0, 2v, 1)$

$\Gamma_u \times \Gamma_v = (-2v, -2u, 4uv)$

$I = \int_0^1 \int_0^1 -2v - 2u + 4uv\,du\,dv$

$= (-2)(\frac{1}{2}) - 2(\frac{1}{2}) + 4(\frac{1}{2})(\frac{1}{2})$

$= -1$ //

(c) $I = \int_S 1 + x^3y^2\,dS$, where S is the surface $z = x^2 + y^2$ above the closed unit disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$.

$$g(x,y) = x^2+y^2, \quad \frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = 2y$$

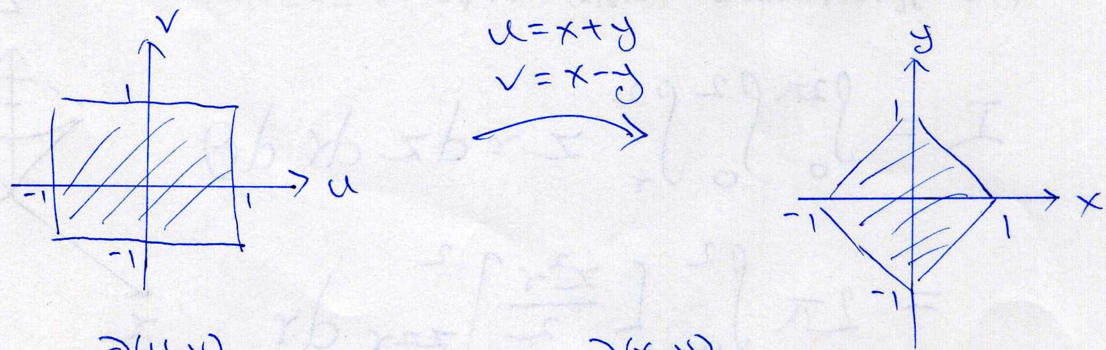
$$I = \int_D (1+x^3y^2)\sqrt{1+(2x)^2+(2y)^2}\,dA$$

$$= \int_D \sqrt{1+4x^2+4y^2}\,dA + 0 \quad (\because \text{odd function})$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\,dr\,d\theta \quad (\text{polar coordinates})$$

$$= \frac{\pi}{6}(5\sqrt{5} - 1) \,_{\shortparallel}$$

(d) $I = \int_{-1}^{1}\int_{|x|-1}^{1-|x|} (x+y)^2\,dy\,dx$. Hint: change of variables $u = x + y$ and $v = x - y$.



$$\frac{\partial(u,v)}{\partial(x,y)} = -2 \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

$$I = \int_{-1}^{1}\int_{-1}^{1} u^2 \left|-\frac{1}{2}\right| du\,dv$$

$$= +\frac{2}{3} \,_{\shortparallel}$$

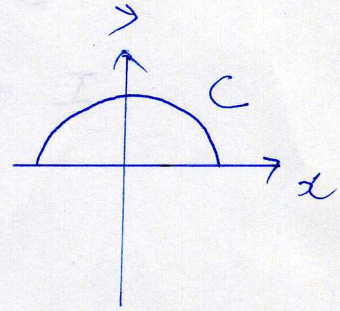
(e) $I = \int_C y\,ds$, where $C = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = 1, y \geq 0\}$ is the upper half of the unit circle.

$\gamma(t) = (\cos t, \sin t)$, $0 \leq t \leq \pi$

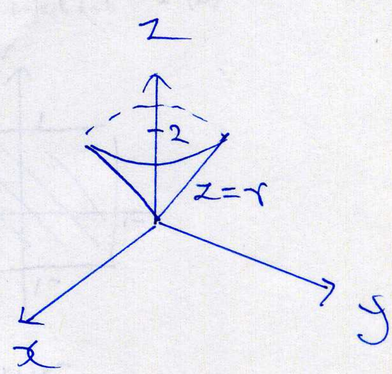
$\gamma'(t) = (-\sin t, \cos t)$

$|\gamma'(t)| = 1$

$I = \int_0^\pi \sin(t)\,dt = 2$ "



(f) $I = \int_E z\,dV$, where $E = \{(x,y,z) \in \mathbb{R}^3 \colon \sqrt{x^2+y^2} \leq z \leq 2\}$.



$I = \int_0^{2\pi} \int_0^2 \int_r^2 z r\,dz\,dr\,d\theta$

$= 2\pi \int_0^2 \left[\frac{z^2 r}{2}\right]_{z=r}^{2} dr$

$= 2\pi \int_0^2 2r - \frac{r^3}{2}\,dr$

$= 4\pi$

Problem 2 (10 points) Consider the vector field $F(x, y) = (2xy, x^2 + 3y^2)$. It is given that F is conservative on $\mathbb{R}^2$.

(a) Find ALL functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ for which $F = \nabla f$.

$$\begin{cases} f_x = 2xy & \text{———— (1)} \\ f_y = x^2 + 3y^2 & \text{———— (2)} \end{cases}$$

Integrate (1) w.r.t. x:

$$f = x^2y + g(y) \quad \text{———— (3)}$$

Differentiate (3) w.r.t. y:

$$f_y = x^2 + g'(y) \quad \text{———— (4)}$$

By (2) and (4):

$$g'(y) = 3y^2 \Rightarrow g(y) = y^3 + C$$

So, $f = x^2y + y^3 + C$,

where C is a constant.

(b) Evaluate $\int_C 2xydx + (x^2 + 3y^2)dy$, where C is a curve from $(1, 2)$ to $(2, 8)$ along the parabola $y = 2x^2$.

$$\int_C 2xydx + (x^2+3y^2)dy$$

$$= f(2,8) - f(1,2) \quad \text{(fundamental theorem of line integral)}$$

$$= 534$$

Problem 3 (10 points) Let $E \subseteq \mathbb{R}^3$ be the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where a, b, $c > 0$ are constants.

(a) Using spherical coordinates, write down a parametrization of E. Hint: E is the image of the unit sphere under the linear map $(x, y, z) \mapsto (ax, by, cz)$.

$$r(\phi, \theta) = (a\sin(\phi)\cos(\theta),\ b\sin(\phi)\sin(\theta),\ c\cos(\phi))$$

$$0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

(b) Setup an iterated integral of the form $\int_a^b \int_c^d f(\phi, \theta)\, d\phi\, d\theta$ which represents the area of E. You don't need to simplify the expression of $f(\phi, \theta)$ or evaluate the integral. Your answer may contain a cross product.

$$\text{area}(E) = \int_0^{2\pi} \int_0^{\pi} |r_\phi \times r_\theta|\, d\phi\, d\theta$$

Problem 4 (10 points) Let $F(x, y) = (y^2, x^2)$.

(a) Is F conservative on $\mathbb{R}^2$? Explain.

$P = y^2, \quad Q = x^2$

$P_y = 2y, \quad Q_x = 2x$

Since $P_y \neq Q_x$, F is not conservative.

(b) Evaluate the curve integral $\int_\gamma F \cdot ds$, where

$$\gamma(t) = \left(t^{t^2+1} \sin\left(\frac{\pi}{6} t^{t^2+1}\right), t^{2(t^2+1)} \sin^2\left(\frac{\pi}{6} t^{t^2+1}\right)\right)$$

is defined for $0 \leq t \leq 1$. Hint: let $u = t^{t^2+1} \sin\left(\frac{\pi}{6} t^{t^2+1}\right)$ and re-parametrize the curve by setting $\gamma_1(u) = (u, u^2) = \gamma(t)$. You may need to figure out the range of u.

$$\int_\gamma F \cdot ds = \int_0^{\frac{1}{2}} F(\gamma_1(u)) \cdot \gamma_1(u)\, du$$

$$= \int_0^{\frac{1}{2}} (u^4, u^2) \cdot (1, 2u)\, du$$

$$= \int_0^{\frac{1}{2}} u^4 + 2u^3\, du$$

$$= \frac{3}{80}$$

Problem 5 (20 points) Let $S = \{(x,y,z) \in \mathbb{R}^3 \colon x^2+y^2+z^2 = 1, y \geq 0\}$ be the upper half of the unit sphere, with outward orientation. That is, the outward unit normal vector field

$$n(x,y,z) = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$$

is chosen for computing surface integral. Let $C = \{(x,y,0) \in \mathbb{R}^3 \colon x^2+y^2 = 1\}$ be oriented counter-clockwise when viewed from above. Let $D = \{(x,y) \in \mathbb{R}^2 \colon x^2+y^2 \leq 1\}$ be the closed unit disk in xy-plane.

(a) Use Stokes' theorem and Green's theorem to show that, for any smooth vector field $F = (P,Q,R)$ defined on $\mathbb{R}^3$,

$$\int_S \operatorname{curl}(F) \cdot dS = \int_D \frac{\partial Q}{\partial x}(x,y,0) - \frac{\partial P}{\partial y}(x,y,0)\, dA.$$

Proof

$$\int_S \operatorname{curl}(F) \cdot dS = \int_C F \cdot ds \quad \text{(Stokes' theorem)}$$

$$= \int_C P\,dx + Q\,dy + \cancel{R\,dz}^{\,0}$$

$$= \int_D \frac{\partial Q}{\partial x}(x,y,0) - \frac{\partial P}{\partial y}(x,y,0)\, dA$$

(Green's theorem)

(b) Evaluate the flux integral $\int_S \operatorname{curl}(F) \cdot dS$, where $F(x,y,z) = (x^2y^3, -2xyz, 3z^2)$.

$$\int_S \operatorname{curl}(F) \cdot dS = \int_D -3x^2y^2\, dA$$

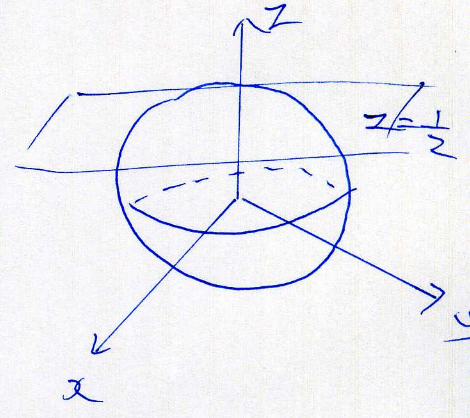
$$= \int_0^{2\pi}\int_0^1 -3r^2\cos^2(\theta)\, r^2\sin^2(\theta)\, r\, dr\, d\theta$$

$$= -3\left(\int_0^{2\pi} \sin^2(\theta)\cos^2(\theta)\, d\theta\right)\left(\int_0^1 r^5\, dr\right)$$

$$= (-3)\left(\frac{\pi}{4}\right)\left(\frac{1}{6}\right)$$

$$= -\frac{\pi}{8} \;\text{//}$$

Problem 6 (20 points) Find the volume of the solid $E = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1, z \ge \frac{1}{2}\}$.



$$\text{Vol}(E) = \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{\frac{1}{2}}^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^{\frac{\sqrt{3}}{2}} r\left(\sqrt{1-r^2} - \frac{1}{2}\right) dr$$

$$= 2\pi \left(\frac{7}{24} - \frac{3}{16}\right)$$

$$= \frac{5\pi}{24} \, //$$