Level Set Methods in Convex Optimization

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Joint work with
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Happy Birthday Andy!

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Workshop on Nonlinear Optimization Algorithms and Industrial Applications
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Motivation

Optimization in Large-Scale Inference

- A range of large-scale data science applications can be modeled using optimization:
  - Inverse problems (medical and seismic imaging)
  - High dimensional inference (compressive sensing, LASSO, quantile regression)
  - Machine learning (classification, matrix completion, robust PCA, time series)
- These applications are often solved using *side information*:
  - Sparsity or low rank of solution
  - Constraints (topography, non-negativity)
  - Regularization (priors, total variation, “dirty” data)
- We need efficient large-scale solvers for *nonsmooth* programs.

I. I. EREMIN, The penalty method in convex programming, Soviet Math. Dokl., 8 (1966), pp. 459-462.

W. I. ZANGWILL, Nonlinear programming via penalty functions, Management Sci., 13 (1967), pp. 344-358.

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R.H. Bartels and A.R. Conn, An Approach to Nonlinear ℓ_1 Data Fitting, Proceedings of the Third Mexican Workshop on Numerical Analysis, pp. 48-58, J. P. Hennart (Ed.), Springer-Verlag, 1981.

The Prototypical Problem

Sparse Data Fitting:

Find sparse x with $Ax \approx b$

There are numerous applications;

- system identification
- image segmentation
- compressed sensing
- grouped sparsity for remote sensor location
- ...

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Convex approaches: $\|x\|_1$ as a sparsity surragate (Candes-Tao-Donaho '05)

BPDN	LASSO	Lagrangian (Penalty)
$\min_x \|x\|_1$ s.t. $\frac{1}{2}\|Ax-b\|_2^2 \leq \sigma$	$\min_x \frac{1}{2}\|Ax-b\|_2^2$ s.t. $\|x\|_1 \leq \tau$	$\min_x \frac{1}{2}\|Ax-b\|_2^2 + \lambda\|x\|_1$

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Basis for SPGL1 (van den Berg-Friedlander '08)

Optimal Value or Level Set Framework

Problem class: Solve

$$\begin{aligned} \min_{x \in \mathcal{X}} \quad & \phi(x) \\ \text{s.t.} \quad & \rho(Ax - b) \le \sigma \end{aligned} \qquad \mathcal{P}(\sigma)$$

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Strategy: Consider the "flipped" problem

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Then opt-val($\mathcal{P}(\sigma)$) is the minimal root of the equation

$$\boxed{v(\tau) = \sigma}$$

Queen Dido's Problem

The intuition behind the proposed framework has a distinguished history, appearing even in antiquity. Perhaps the earliest instance is Queen Dido's problem and the fabled origins of Carthage.

In short, the problem is to find the maximum area that can be enclosed by an arc of fixed length and a given line. The converse problem is to find an arc of least length that traps a fixed area between a line and the arc. Although these two problems reverse the objective and the constraint, the solution in each case is a semi-circle.

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Other historical examples abound (e.g. the isoperimetric inequality). More recently, these observations provide the basis for the Markowitz Mean-Variance Portfolio Theory.

The Role of Convexity

Convex Sets

Let $C \subset \mathbb{R}^n$. We say that C is convex if

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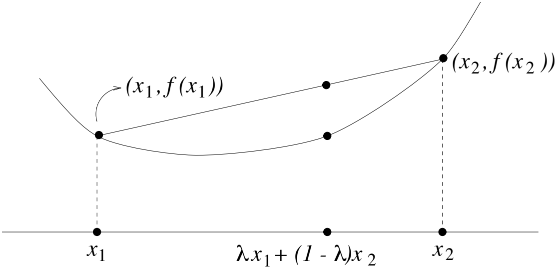

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2)$$

Convex indicator functions

Let $C \subset \mathbb{R}^n$. Then the function

$$\delta_C(x) := \begin{cases} 0 & \text{, if } x \in C, \\ +\infty & \text{, if } x \notin C, \end{cases}$$

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Addition

Non-negative linear combinations of convex functions are convex: f_i convex and $\lambda_i \geq 0, i = 1, \ldots, k$

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Infimal Projection

If $f : \mathbb{R}^n \times \mathbb{R}^m \to \bar{\mathbf{R}}$ is convex, then so is

$$v(x) := \inf_y f(x, y),$$

since

$$\operatorname{epi}(v) = \{\, (x, \mu) \,:\, \exists\, y \in \text{ s.t. } f(x, y) \leq \mu \,\}.$$

Convexity of v

When $\mathcal{X}$, ρ, and ϕ are convex, the optimal value function v is a non-increasing convex function by infimal projection:

$$v(\tau) := \min_{x \in \mathcal{X}} \quad \rho(Ax - b) \quad \text{s.t.} \quad \phi(x) \le \tau$$

$$= \min_{x} \quad \rho(Ax - b) + \delta_{\text{epi}\,(\phi)}(x, \tau) + \delta_{\mathcal{X}}(x)$$

Newton and Secant Methods

For f convex and non-increasing, solve $f(\tau) = 0$.

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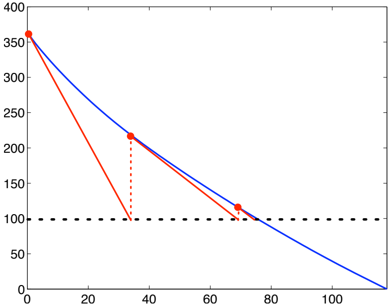

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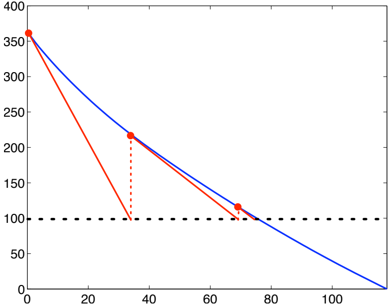

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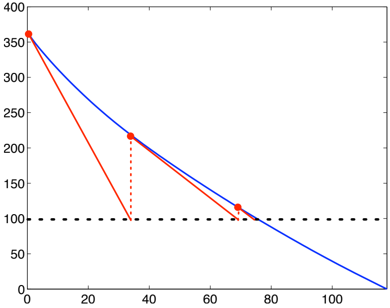

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Use the convex subdifferential

$$\partial f(x) := \{\, z \,:\, f(y) \geq f(x) + z^T(y - x) \quad \forall\ y \in \mathbb{R}^n \,\}$$

Superlinear Convergence

Let $\tau_* := \inf\{\tau : f(\tau) \le 0\}$ and assume

$$g_* := \inf \{ \, g \; : \; g \in \partial f(\tau_*) \, \} < 0 \qquad \text{(non-degeneracy)}$$

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Initialization: $\tau_{-1} < \tau_0 < \tau_*$

$$\tau_{k+1} := \begin{cases} \tau_k & \text{if } f(\tau_k) = 0, \\ \tau_k - \frac{f(\tau_k)}{g_k} \qquad [\text{for } g_k \in \partial f(\tau_k)] & \text{otherwise;} \end{cases} \qquad \text{(Newton)}$$

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If either sequence terminates finitely at some τ_k, then $\tau_k = \tau_*$; otherwise,

$$|\tau_* - \tau_{k+1}| \le (1 - \frac{g_*}{\gamma_k})|\tau_* - \tau_k|, \;\; k = 1, 2, \dots,$$

where $\gamma_k = g_k$ (Newton) and $\gamma_k \in \partial f(\tau_{k-1})$ (secant). In either case, $\gamma_k \uparrow g_*$ and $\tau_k \uparrow \tau_*$ globally q-superlinearly.

Inexact Root Finding

- Problem: Find root of the inexactly known convex function

$$v(\cdot) - \sigma.$$

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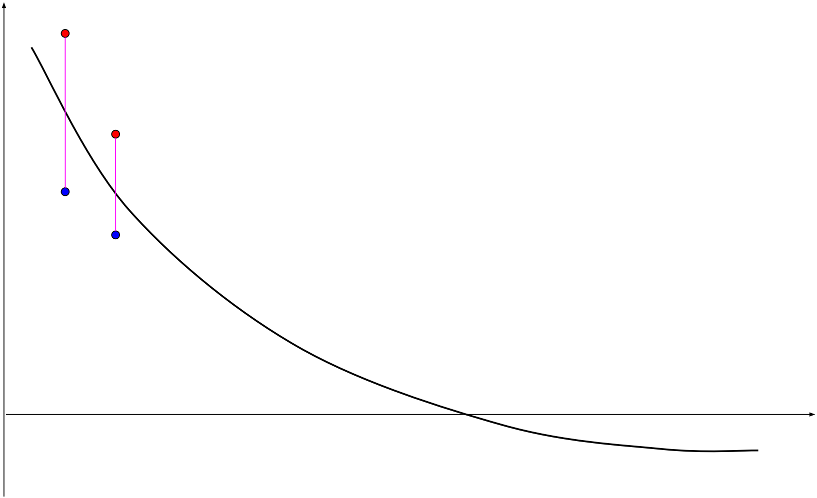
- Solution:
  - modified secant
  - approximate Newton methods

Inexact Root Finding: Secant



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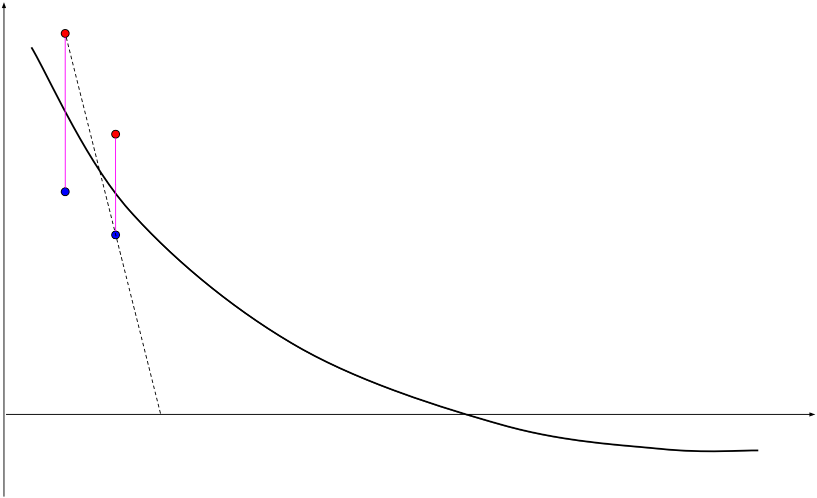

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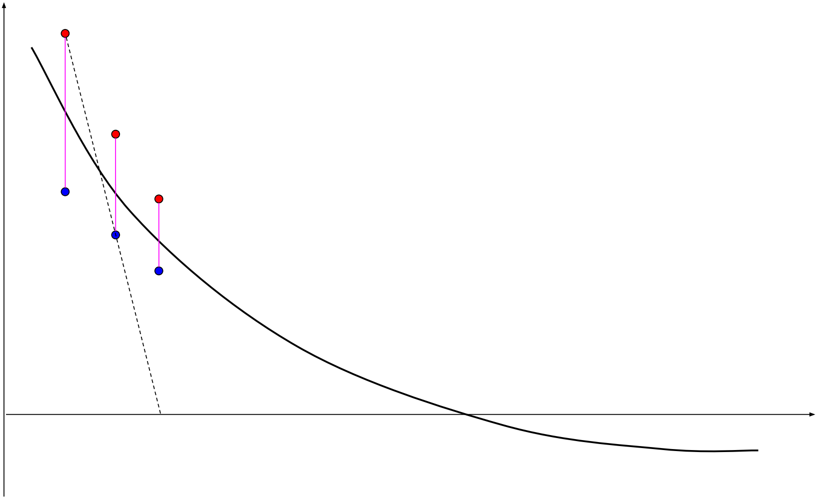

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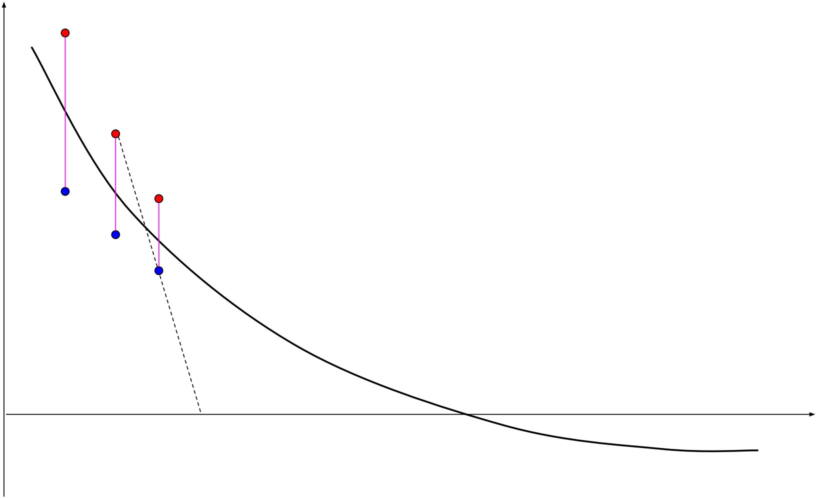

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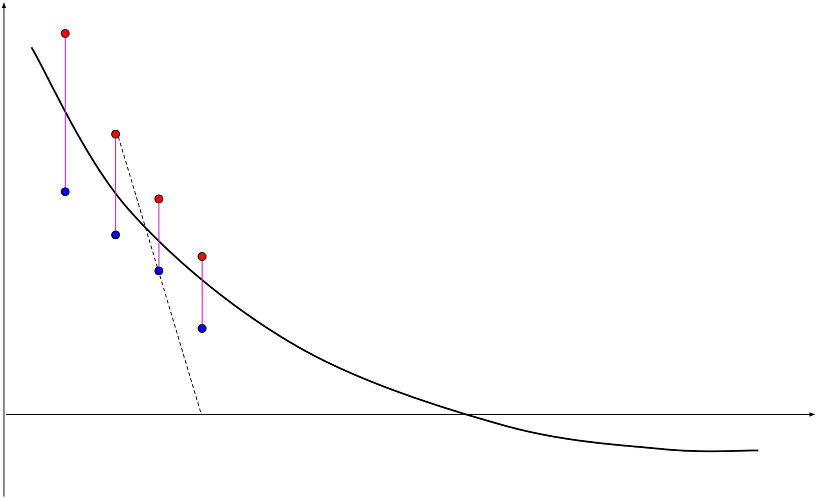

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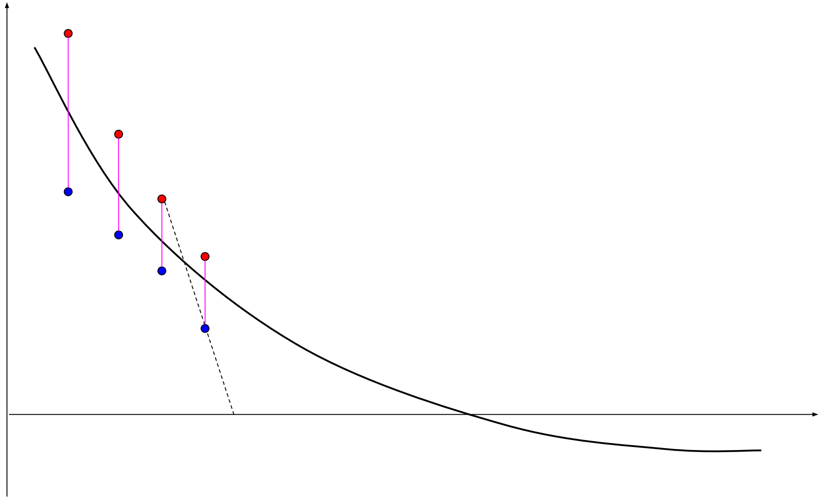
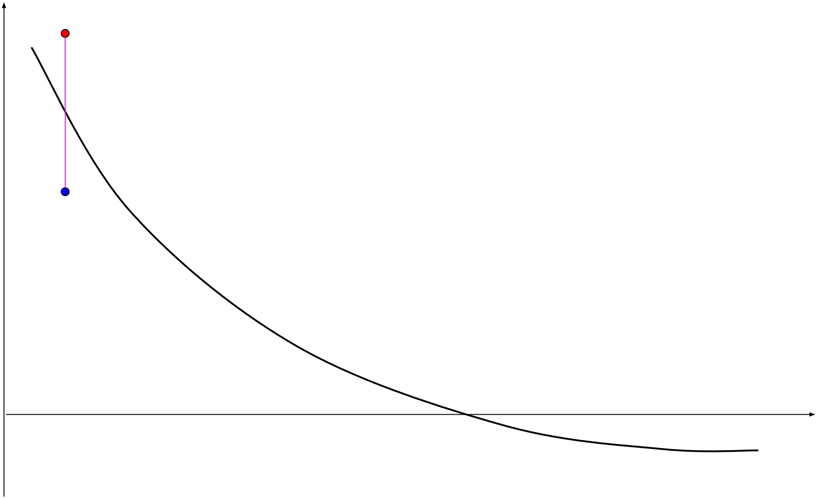


Inexact Root Finding: Newton



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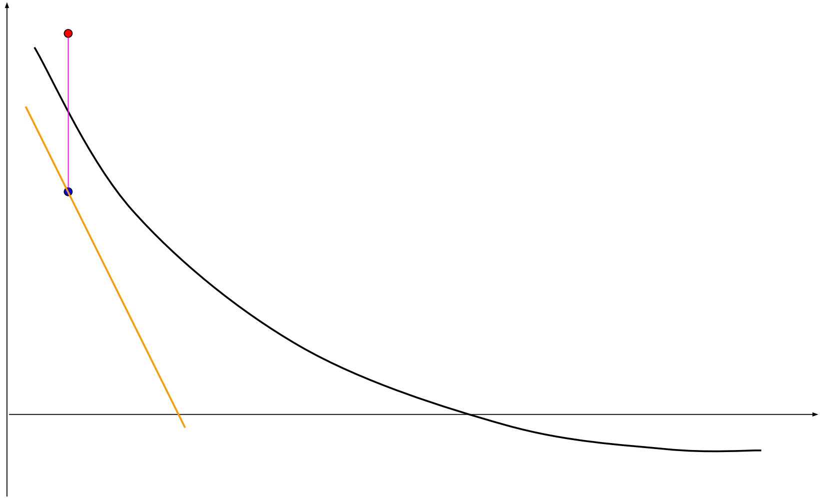

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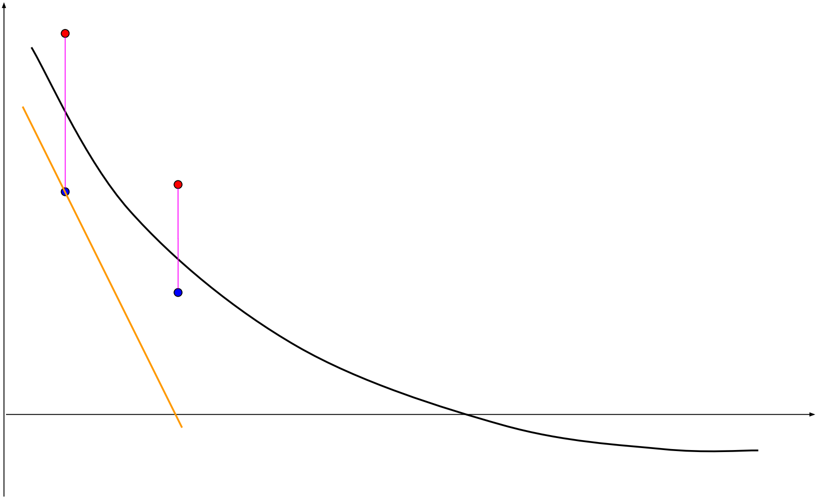

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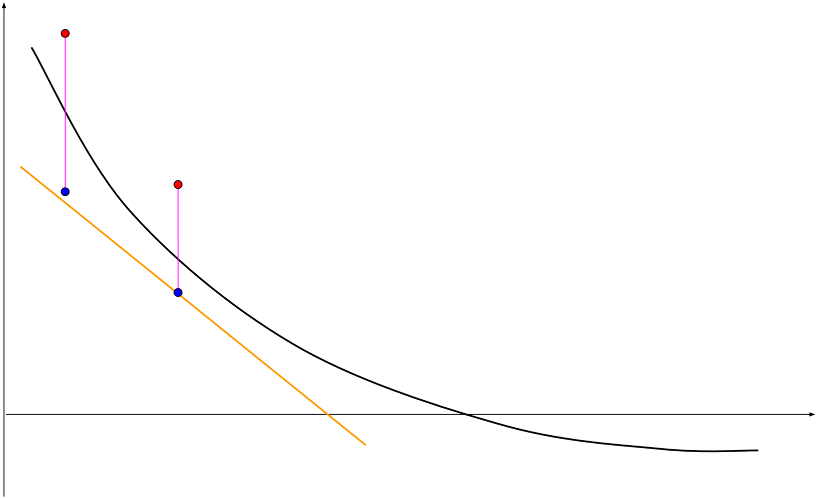

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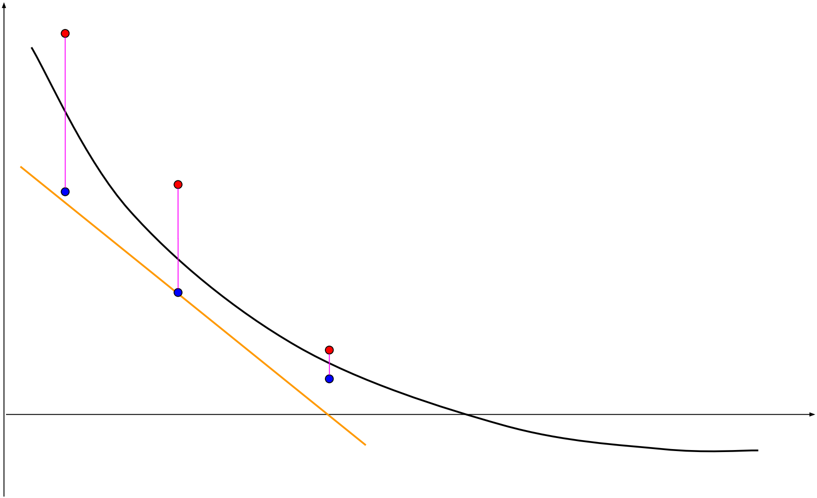

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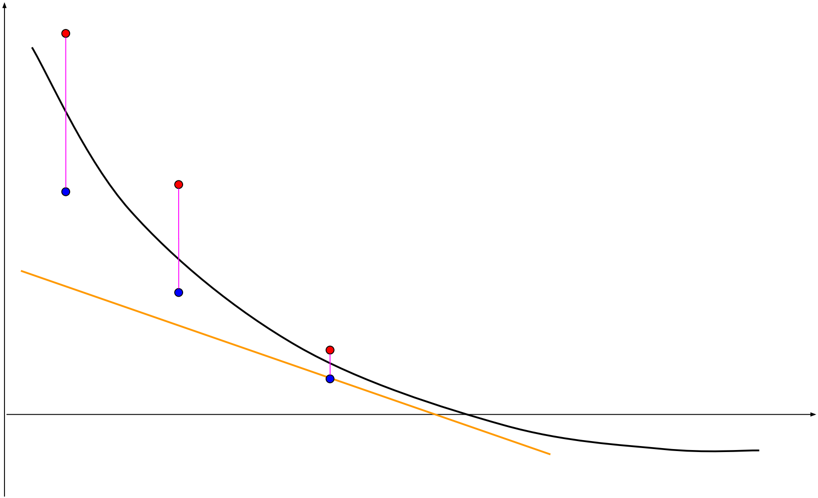
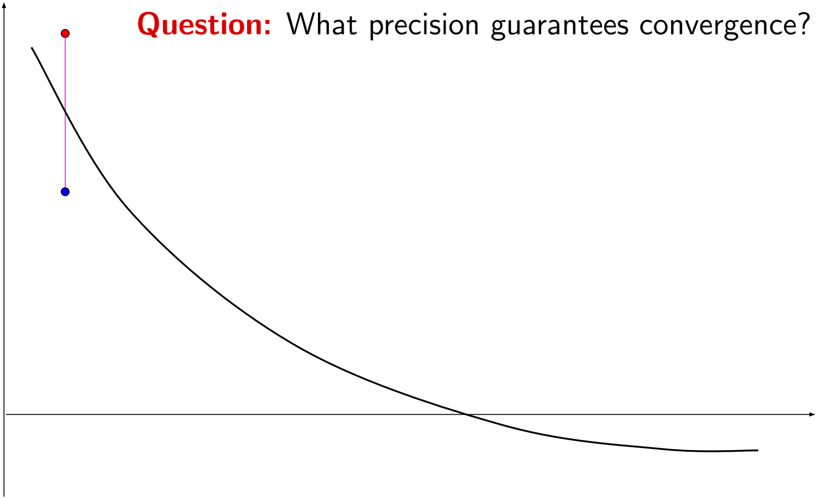


Inexact Root Finding: Convergence

Question: What precision guarantees convergence?



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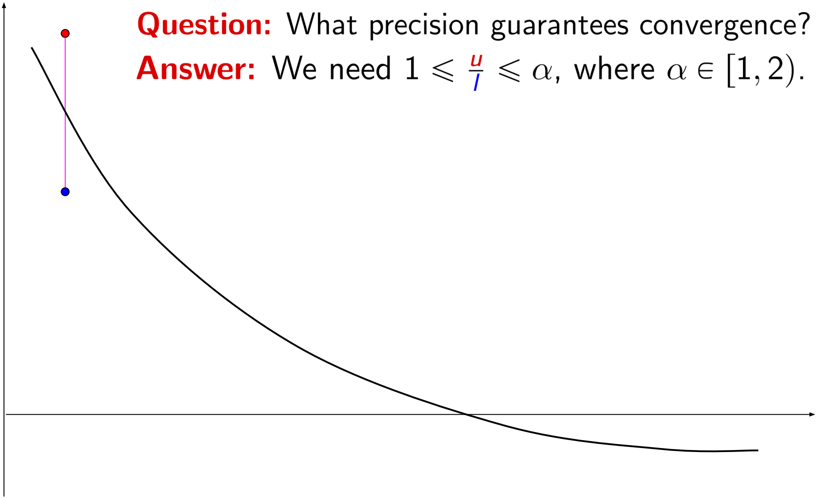
Answer: We need $1 \leqslant \frac{u}{l} \leqslant \alpha$, where $\alpha \in [1, 2)$.



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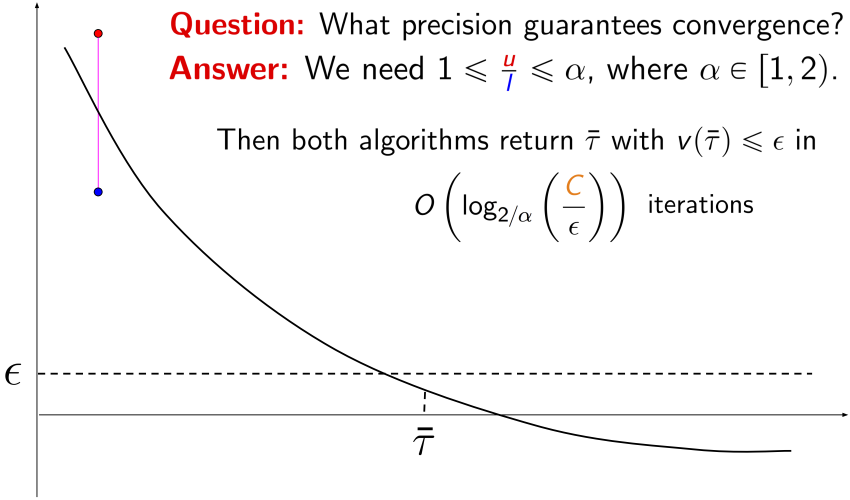

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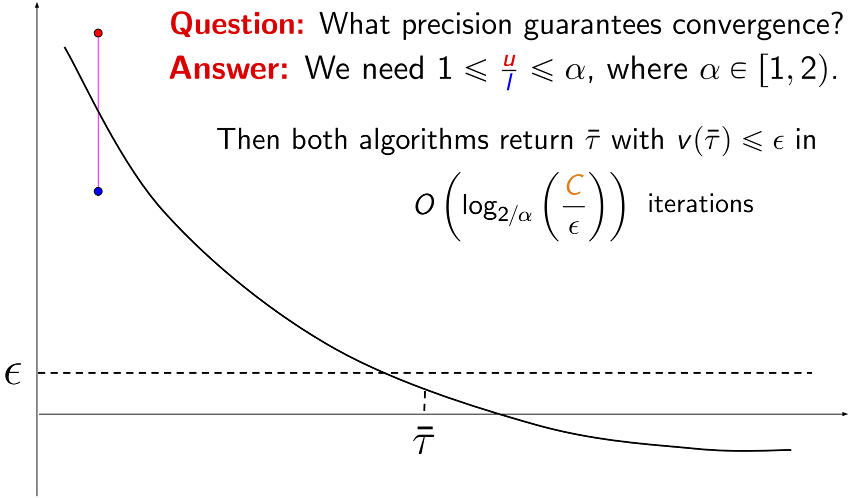
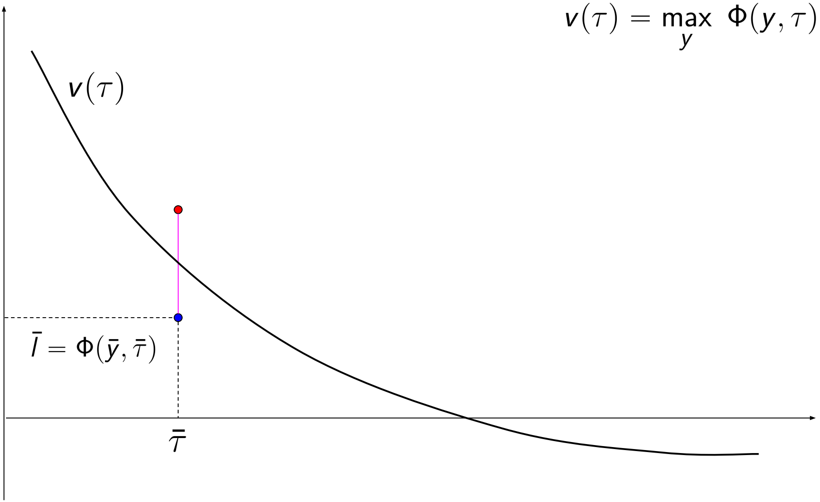


Key observation: $C = C(\tau_0)$ is independent of $\partial v(\tau^*)$.
Nondegeneracy not required.

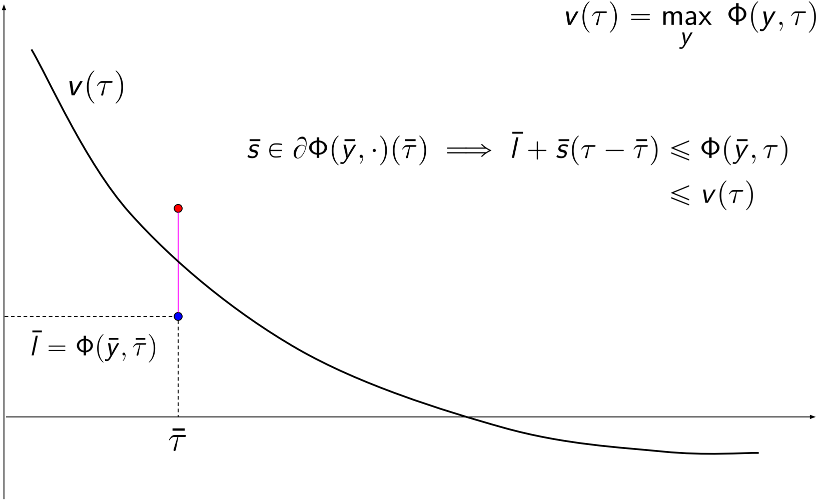
Minorants from Duality



Minorants from Duality

$$v(\tau) = \max_y \; \Phi(y, \tau)$$

$$\bar{s} \in \partial \Phi(\bar{y}, \cdot)(\bar{\tau}) \implies \bar{l} + \bar{s}(\tau - \bar{\tau}) \leqslant \Phi(\bar{y}, \tau) \leqslant v(\tau)$$



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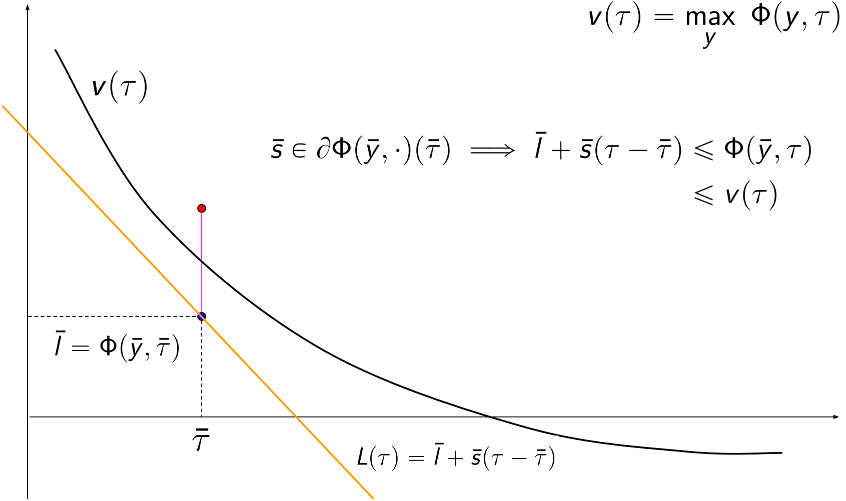
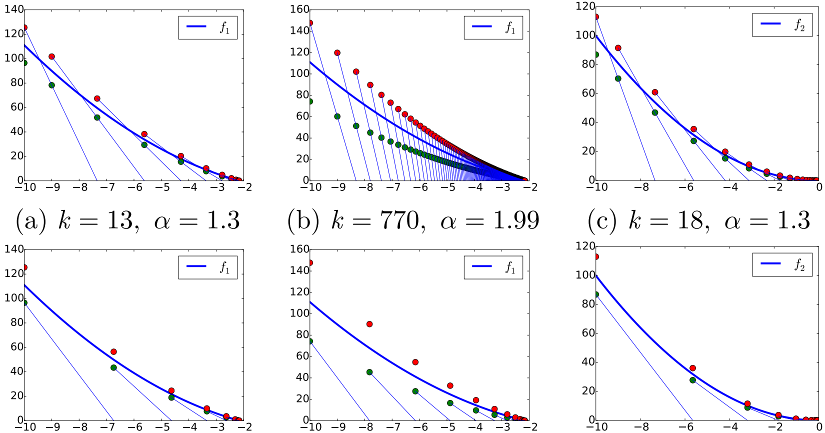


Robustness: $1 \leq u/l \leq \alpha$, where $\alpha \in [1, 2)$ and $\epsilon = 10^{-2}$



(a) $k = 13,\ \alpha = 1.3$ (b) $k = 770,\ \alpha = 1.99$ (c) $k = 18,\ \alpha = 1.3$

(d) $k = 9,\ \alpha = 1.3$ (e) $k = 15,\ \alpha = 1.99$ (f) $k = 10,\ \alpha = 1.3$

Figure : Inexact secant (top) and Newton (bottom) for $f_1(\tau) = (\tau - 1)^2 - 10$ (first two columns) and $f_2(\tau) = \tau^2$ (last column). Below each panel, α is the oracle accuracy, and k is the number of iterations needed to converge, i.e., to reach $f_i(\tau_k) \leq \epsilon = 10^{-2}$.

When is the Level Set Framework Viable

Problem class: Solve

$$\min_{x \in \mathcal{X}} \quad \phi(x) \qquad \text{s.t.} \quad \rho(Ax - b) \le \sigma \qquad \mathcal{P}(\sigma)$$

Strategy: Consider the “flipped” problem

$$v(\tau) := \min_{x \in \mathcal{X}} \quad \rho(Ax - b) \qquad \text{s.t.} \quad \phi(x) \le \tau \qquad \mathcal{Q}(\tau)$$

Then opt-val($\mathcal{P}(\sigma)$) is the minimal root of the equation

$$\boxed{v(\tau) = \sigma}$$

Lower bounding slopes: $\partial_\tau \Phi(y, \tau)$

Convex Indicator

For any convex set C, the convex indicator function for C is

$$\delta\left(x \mid C\right) := \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

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Fact If $0 \in C$, then $\gamma\left(x \mid C\right) = \delta^*\left(x \mid C^\circ\right)$, where

$$C^\circ := \left\{z \mid \langle z, x\rangle \leq 1 \ \forall\, x \in C\right\}.$$

Gauge Optimization

Problem	$\mathcal{P}_\sigma$	$\mathcal{Q}_\tau$	$\partial_\tau \Phi(y,\tau)$
gauge optimization	$\min_x \ \varphi(x)$ s.t. $\rho(Ax-b) \le \sigma$	$\min_x \ \rho(Ax-b)$ s.t. $\varphi(x) \le \tau$	$-\varphi^\circ(A^T y)$
BPDN	$\min_x \ \\|x\\|_1$ s.t. $\\|Ax-b\\|_2 \le \sigma$	$\min_x \ \\|Ax-b\\|_2$ s.t. $\\|x\\|_1 \le \tau$	$-\\|A^T y\\|_\infty$
sharp elast-net	$\min_x \ \alpha\\|x\\|_1 + \beta\\|x\\|_2$ s.t. $\\|Ax-b\\|_2 \le \sigma$	$\min_x \ \\|Ax-b\\|_2$ s.t. $\alpha\\|x\\|_1 + \beta\\|x\\|_2 \le \tau$	$-\gamma_{\alpha\mathbb{B}_\infty+\beta\mathbb{B}_2}(A^T y)$
matrix completion	$\min_X \ \\|X\\|_*$ s.t. $\\|\mathcal{A}X-b\\|_2 \le \sigma$	$\min_x \ \\|\mathcal{A}X-b\\|_2$ s.t. $\\|X\\|_* \le \tau$	$-\sigma_1(\mathcal{A}^* y)$

Nonsmooth regularized data-fitting.

Piecewise Linear-Quadratic Penalties

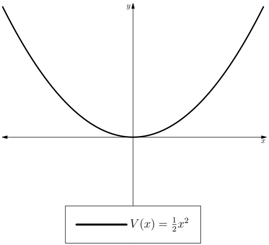
$$\phi(x) := \sup_{u \in U} [\langle x, u \rangle - \frac{1}{2} u^T B u]$$

$U \subset \mathbb{R}^n$ is nonempty, closed and convex with $0 \in U$ (not nec. poly.)
$B \in \mathbb{R}^{n \times n}$ is symmetric positive semi-definite.

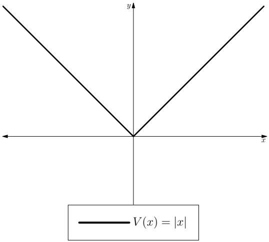
Examples:

1. Support functionals: $B = 0$
2. Gauge functionals: $\gamma(\cdot \mid U^\circ) = \delta^*(\cdot \mid U)$
3. Norms: $\mathbb{B}$ = closed unit ball, $\|\cdot\| = \gamma(\cdot \mid \mathbb{B})$
4. Least-squares: $U = \mathbb{R}^n$, $B = I$
5. Huber: $U = [-\epsilon, \epsilon]^n$, $B = I$

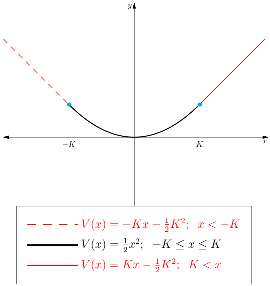
PLQ Densities: Gauss, Laplace, Huber, Vapnik



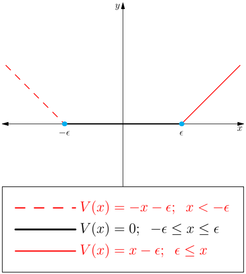
Gauss



ℓ_1



Huber



Vapnik

$\partial_\tau \Phi(y,\tau)$ for PLQ Penalties ϕ

$$\phi(x) := \sup_{u \in U} \left[\langle x, u\rangle - \frac{1}{2} u^T B u\right]$$

$$\mathcal{P}_\sigma \qquad \min \phi(x) \quad \text{st } \rho(b - Ax) \le \sigma$$

$$\mathcal{Q}_\tau \qquad \min \rho(b - Ax) \quad \text{st } \phi(x) \le \tau$$

$$-\max\left\{\gamma\left(A^T y \mid U\right),\ \sqrt{y^T A B A^T y}/\sqrt{2\tau}\right\} \in \partial_\tau \Phi(y,\tau)$$

Sparse and Robust Formulation

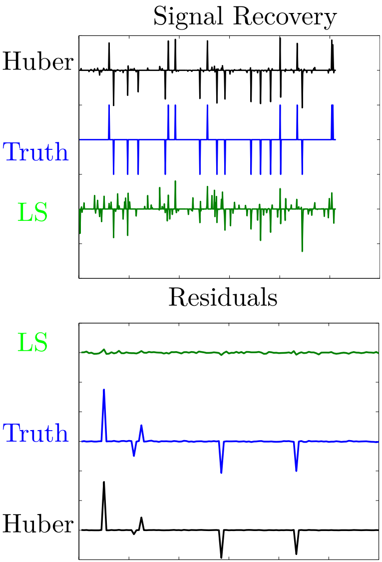
$$\text{HBP}_\sigma: \quad \min \quad \|x\|_1 \quad \text{st} \quad \rho(b - Ax) \le \sigma$$

Problem Specification

- x 20-sparse spike train in $\mathbb{R}^{512}$
- b measurements in $\mathbb{R}^{120}$
- A Measurement matrix satisfying RIP
- ρ Huber function
- σ error level set at .01
- 5 outliers

Results

In the presence of outliers, the robust formulation recovers the spike train, while the standard formulation does not.



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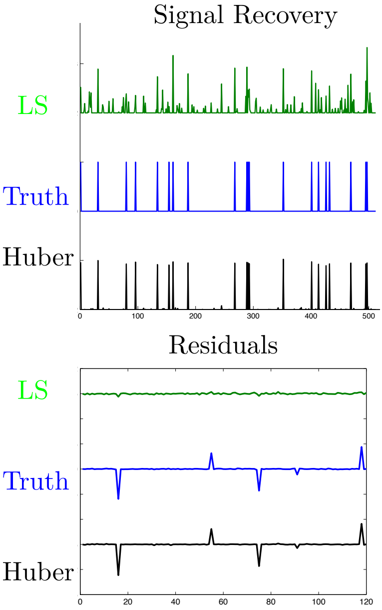
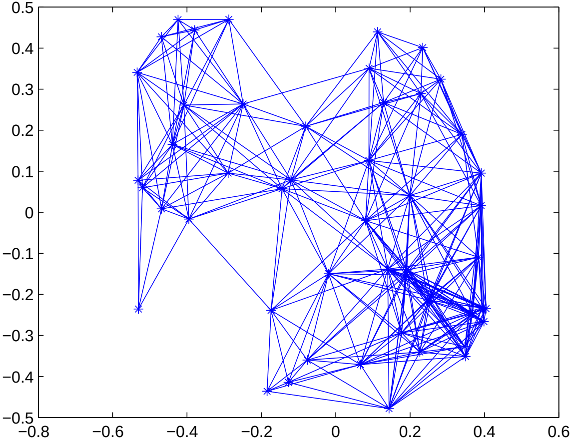


Sensor Network Localization (SNL)



Given a weighted graph $G = (V, E, d)$ find a **realization**:

$$p_1, \ldots, p_n \in \mathbf{R}^2 \quad \text{with} \quad d_{ij} = \|p_i - p_j\|^2 \quad \text{for all } ij \in E.$$

Sensor Network Localization (SNL)

SDP relaxation (Weinberger et al. '04, Biswas et al. '06):

$$\begin{aligned} \max \quad & \operatorname{tr}(X) \\ \text{s.t.} \quad & \|\mathcal{P}_E \mathcal{K}(X) - d\|_2^2 \le \sigma \\ & Xe = 0, \quad X \succeq 0 \end{aligned}$$

where $[\mathcal{K}(X)]_{i,j} = X_{ii} + X_{jj} - 2X_{ij}$.

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Intuition: $X = PP^T$ and then $\operatorname{tr}(X) = \frac{1}{n+1} \sum_{i,j=1}^{n} \|p_i - p_j\|^2$
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Key point: Slater failing (always the case) is irrelevant.

Approximate Newton

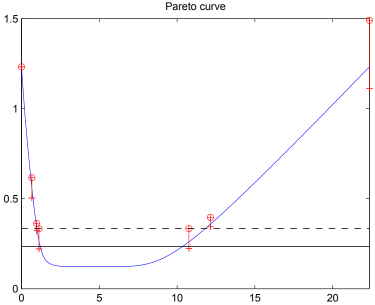
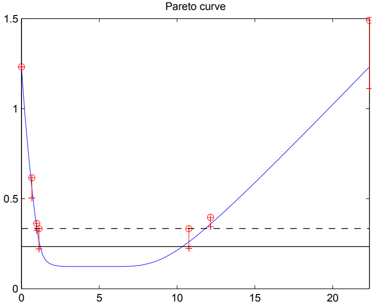


Figure : $\sigma = 0.25$

Approximate Newton



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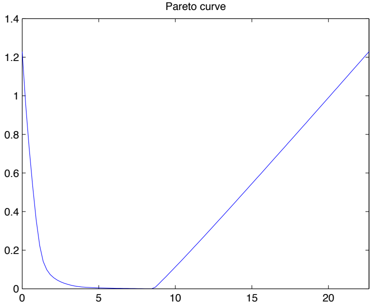
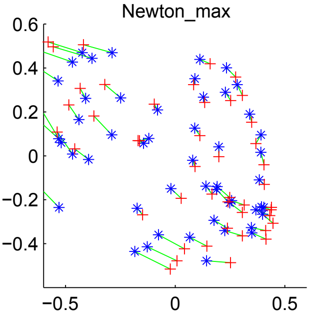
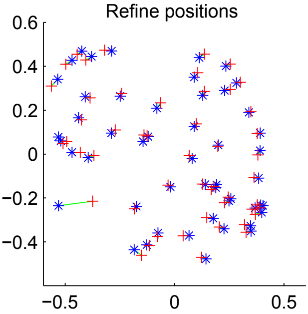


Figure : $\sigma = 0$

Max-trace

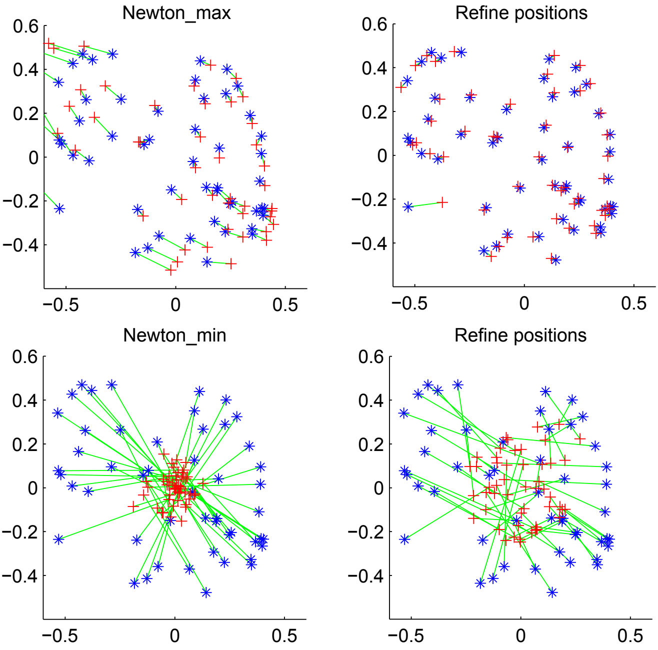


Newton_max
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5



Refine positions
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5

Max-trace



Newton_max
Refine positions
Newton_min
Refine positions
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5

Observations

- Simple strategy for optimizing over complex domains
- Rigorous convergence guarantees
- Insensitivity to ill-conditioning
- Many applications
  - Sensor Network Localization (Drusvyatskiy-Krislock-Voronin-Wolkowicz '15)
  - Sparse/Robust Estimation and Kalman Smoothing (Aravkin-B-Pillonetto '13)
  - Large scale SDP and LP (cf. Renegar '14)
  - Chromosome reconstruction (Aravkin-Becker-Drusvyatskiy-Lozano '15)
  - Phase retrieval (Aravkin-B-Drusvyatskiy-Friedlander-Roy '16)
  - Generalized linear models (Aravkin-B-Drusvyatskiy-Friedlander-Roy '16)
  - ...

Thank you!

Thank you!

Andy

Thank you!

Andy and Barbara

References

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  Preprint, 2016

General Level Set Theorem

$\psi_i : X \subseteq \mathbb{R}^n \to \overline{\mathbb{R}},\ i = 1, 2,$ arbitrary functions and X an arbitrary set.

$$\text{epi}\,(\psi) := \{\,(x, \mu)\,\}\,\psi(x) \leq \mu$$

$$v_1(\sigma) := \inf_{x \in X}\ \psi_1(x) + \delta\left((x, \sigma)\ |\,\text{epi}\,(\psi_2)\right) \qquad \mathcal{P}_{1,2}(\sigma)$$

$$v_2(\tau) := \inf_{x \in X}\ \psi_2(x) + \delta\left((x, \tau)\ |\,\text{epi}\,(\psi_1)\right) \qquad \mathcal{P}_{2,1}(\tau)$$

$$\mathcal{S}_{1,2} := \{\,\sigma \in \overline{\mathbb{R}}\,\}\,\emptyset \neq \operatorname{argmin} \mathcal{P}_{1,2}(\sigma) \subset \{\,x \in X\,\}\,\psi_2(x) = \sigma$$

Then, for every $\sigma \in \mathcal{S}_{1,2}$,

(a) $v_2(v_1(\sigma)) = \sigma$, and

(b) $\operatorname{argmin} \mathcal{P}_{1,2}(\sigma) = \operatorname{argmin} \mathcal{P}_{2,1}(v_1(\sigma)) \subset$
$\{\,x \in X\,\}\,\psi_1(x) = v_1(\sigma)$.

Moreover, $\mathcal{S}_{2,1} = \{\,v_1(\sigma)\,\}\,\sigma \in \mathcal{S}_{1,2}$ and

$$\{\,(\sigma, v_1(\sigma))\,\}\,\sigma \in \mathcal{S}_{1,2} \ = \ \{\,(v_2(\tau), \tau)\,\}\,\tau \in \mathcal{S}_{2,1}.$$