The exam will consist of 7 questions each worth 50 points for a total of 350 points.. The final is comprehensive. The content of each question is listed below. These questions may be multipart. You must show all of your work to get full credit for your solution. Except in the case of the True-False questions, an answer that appears with no accompanying work will be given zero credit.

Question 1: (50 points) In this question you will be asked to give a complete description of the solution set to one or more linear systems of equations by using row operations to convert the associated augmented matrices to reduced echelon form.

Question 2: (50 points) In this question you will be asked to compute a basis for one or more of the 4 fundamental subspaces associated with a matrix A: $\mathrm{Ran}(A)$, $\mathrm{Nul}(A)$, $\mathrm{Ran}(A^T)$, $\mathrm{Nul}(A^T)$. The subspace may be described as either the linear span of a collection of vectors or the subspace orthogonal to a collection of vectors.

Question 3: (50 points) In this question you will be asked to apply the Gram-Schmidt orthogonalization process to compute and orthogonal or orthonormal basis for a subspace. The subspace may be specified as one of the four fundamental subspaces associated with a matrix.

Question 4: (50 points) This problem concerns linear least-squares problems. For example, you may be asked to solve one or more linear least-squares problems, or you may be asked to solve a problem of fitting the coefficients of a polynomial from data.

Question 5: (50 points) This question concerns determinants. For example, you may be asked to compute the determinant of a matrix, or matrices, or you may be asked to use a property of determinants to answer a question about a given matrix.

Question 6: (50 points) This question concerns eigenvalues and eigenvectors. You may be asked to compute them, or you may be asked to use them to answer a related question.

Question 7: (50 points) This question is a collection of true-false questions based on the the material covered in the text.

SAMPLE QUESTIONS

1. Describe the solution set the following linear system in vector form by converting it to reduced echelon form:

$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & - & x_4 & + & 2x_5 & = & 2 \\ & & -x_2 & + & x_3 & + & x_4 & - & x_5 & = & 3 \\ x_1 & + & 3x_2 & + & x_3 & & & + & 3x_5 & = & -1 \end{array} .$$

2. Compute a basis for the 4 fundamental subspaces associated with the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ 2 & 1 & 3 & -1 & 1 \\ 3 & -1 & 4 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ -1 & 1 & -4 \\ 3 & 5 & 4 \end{bmatrix} . \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 3 & 4 & 5 \\ 1 & 5 & -5 & 1 \end{bmatrix}$$

3. Compute an orthogonal basis for $\mathrm{Ran}(A^T)^{\perp}$ and for $\mathrm{Ran}(A^T)$ where

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & -2 \\ 2 & 1 & 7 & 0 \\ 1 & -2 & 1 & 5 \end{bmatrix} .$$

4. Find the least-squares quadratic fit to the following data sets:

t	-2	-1	1	2
y	2	0	1	2

t	-2	-1	0	1
y	-3	-1	0	3

.

5. Compute the determinants of the following matrices:
Hint: It is far simpler if you use the rules of the determinant rather than Laplaces formula.

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 & 2 & 6 \\ 1 & 3 & 2 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & 2 & 1 & 1 \end{bmatrix}.$$

6. (a) Consider the matrix

$$A = \begin{bmatrix} 3 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

(a) What is the characteristic polynomial for A?
(b) What are the eigenvalues of A?
(c) Without computing eigenvectors, explain why A is nondefective.

(b) The matrix

$$A = \begin{bmatrix} -6 & -1 & 2 \\ 3 & 2 & 0 \\ -14 & -2 & 5 \end{bmatrix}$$

has characteristic polynomial $p(t) = -(t-1)^2(t+1)$. Determine whether A is defective or nondefective (You must show your work!).

(c) Consider the matrix

$$A = \begin{bmatrix} 0 & .3 \\ .6 & 1+\alpha \end{bmatrix}$$

and initial vector $x_0 = [1,\ 1]^T$.

i. For what values of α does the matrix A have an eigenvalue of 1?
ii. For the value of α computed in Part (a), compute the limit of the sequence of vectors given by the iteration $x_{k+1} = Ax_k$ for $k = 0, 1, 2, \ldots$ with initial vector x_0.

7. Answer the following true-false questions.

(T)—(F) : A linear system can have precisely 3 distinct solutions.

(T)—(F) : The matrix $M = \begin{bmatrix} b & 3 \\ b^2 & a \end{bmatrix}$ is nonsingular if and only if either $b = 0$ or $a = 3b$.

(T)—(F) : There exists a matrix A in $\mathbb{R}^{3\times 4}$ such that $\mathrm{Nul}\,(A) = \{0\}$.

(T)—(F) : If $A, B \in \mathbb{R}^{n\times n}$ are invertible, then $A + B$ is invertible.

(T)—(F) : Every matrix satisfying $H^2 = 5H$ is singular.

(T)—(F) : A homogeneous linear system of 5 equations and 6 unknowns always has a non-trivial solution.

(T)—(F) : If $B \in \mathbb{R}^n$ satisfies $B^3 + B^2 + B = I$, then B is nonsingular.

Ⓣ—Ⓕ : The rank of a matrix is equal to the number of its non-zero rows.
Ⓣ—Ⓕ : If the matrix A is row equivalent to B, then B is row equivalent to A.
Ⓣ—Ⓕ :
Ⓣ—Ⓕ : Row equivalent matrices always have the same reduced echelon form.
Ⓣ—Ⓕ : There are 2^3 subspaces in $I\!R^2$.
Ⓣ—Ⓕ : Let $A, B \in I\!R^{n\times n}$. If B is nonsingular and AB is nonsingular, then A is nonsingular.
Ⓣ—Ⓕ : Let $A \in I\!R^{n\times n}$. If A^T is singular, the A cannot have an inverse.
Ⓣ—Ⓕ : If $B^2 + B + 2I = 0$, then B is singular.
Ⓣ—Ⓕ : If u, v, w are any vectors in $I\!R^4$, then the vectors $3u - 2v,\ 2v - 4u$, and $4w - 3u$ are linearly dependent.
Ⓣ—Ⓕ : If $A = BC$, then every solution to $Cx = 0$ is a solution to $A = 0$.
Ⓣ—Ⓕ : Every set of 6 vectors in a 4 dimensional subspace of $I\!R^{12}$ is linearly dependent.
Ⓣ—Ⓕ : If $B = \{v^1, v^2, \ldots, v^n\}$ is a basis for $I\!R^n$ and W is a subspace of $I\!R^n$, then some subset of B is a basis for W.
Ⓣ—Ⓕ : The set of solutions to a linear system of equations form a subspace.
Ⓣ—Ⓕ : If W_1 and W_2 are subspaces of $I\!R^n$ of dimension $k < n$ with $W_1 \subset W_2$, then $W_1 = W_2$.
Ⓣ—Ⓕ : If $\{v^1, v^2, v^3\}$ are linearly independent vectors in $I\!R^5$ and $A \in I\!R^{5\times 5}$ is nonsingular, then the vectors $\{Av^1,\ Av^2,\ Av^3\}$ are linearly independent.
Ⓣ—Ⓕ : If $A, B \in I\!R^{n\times n}$ are such that one of them is singular, then both of the matrices AB and BA must be singular.
Ⓣ—Ⓕ : There exits a 3×2 matrix A and a 2×3 matrix B such that AB is the identity matrix.
Ⓣ—Ⓕ : If x is an eigenvector for A, then x is also an eigenvector for A^T.
Ⓣ—Ⓕ : If A is defective, then A is singular.
Ⓣ—Ⓕ : If $A^5 = 0$, then $\lambda = 0$ is an eigenvalue of A.
Ⓣ—Ⓕ : If $\lambda = 2$ is an eigenvalue for $A \in I\!R^{n\times n}$, then the matrix $A^2 - 3A + 2I_n$ is singular.
Ⓣ—Ⓕ : If $A^2 - 4I_2$ is singular, then $\lambda = -2$ is an eigenvalue of A.
Ⓣ—Ⓕ : If $A \in I\!R^{2\times 2}$ is such that $(A - 2I_2)^2 = 0$, then the only eigenvector of A is $\lambda = 2$ and A is not defective.