Working with Block Structured Matrices

Numerical linear algebra lies at the heart of modern scientific computing and computational science. Today it is not uncommon to perform numerical computations with matrices having millions of components. The key to understanding how to implement such algorithms is to exploit underlying structure within the matrices. In these notes we touch on a few ideas and tools for dissecting matrix structure. Specifically we are concerned with block matrix structures.

1. Rows and Columns

Let $A \in \mathrm{R}^{m\times n}$ so that A has m rows and n columns. Denote the element of A in the ith row and jth column as A_{ij}. Denote the m rows of A by $A_{1\cdot}, A_{2\cdot}, A_{3\cdot}, \ldots, A_{m\cdot}$ and the n columns of A by $A_{\cdot 1}, A_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}$. For example, if

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 & 7 & 3 \\ -2 & 27 & 32 & -100 & 0 & 0 \\ -89 & 0 & 47 & 22 & -21 & 33 \end{bmatrix},$$

then $A_{2,4} = -100$,

$A_{1\cdot} = \begin{bmatrix} 3 & 2 & -1 & 5 & 7 & 3 \end{bmatrix}$, $A_{2\cdot} = \begin{bmatrix} -2 & 27 & 32 & -100 & 0 & 0 \end{bmatrix}$, $A_{3\cdot} = \begin{bmatrix} -89 & 0 & 47 & 22 & -21 & 33 \end{bmatrix}$

and

$$A_{\cdot 1} = \begin{bmatrix} 3 \\ -2 \\ -89 \end{bmatrix}, \ A_{\cdot 2} = \begin{bmatrix} 2 \\ 27 \\ 0 \end{bmatrix}, \ A_{\cdot 3} = \begin{bmatrix} -1 \\ 32 \\ 47 \end{bmatrix}, \ A_{\cdot 4} = \begin{bmatrix} 5 \\ -100 \\ 22 \end{bmatrix}, \ A_{\cdot 5} = \begin{bmatrix} 7 \\ 0 \\ -21 \end{bmatrix}, \ A_{\cdot 6} = \begin{bmatrix} 3 \\ 0 \\ 33 \end{bmatrix}.$$

Exercise: If

$$C = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix},$$

what are $C_{4,4}$, $C_{\cdot 4}$ and $C_{4\cdot}$? For example, $C_{2\cdot} = \begin{bmatrix} 2 & 2 & 0 & 0 & 1 & 0 \end{bmatrix}$ and $C_{\cdot 2} = \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

The block structuring of a matrix into its rows and columns is of fundamental importance and is extremely useful in understanding the properties of a matrix. In particular, for $A \in \mathrm{R}^{m\times n}$ it allows us to write

$$A = \begin{bmatrix} A_{1\cdot} \\ A_{2\cdot} \\ A_{3\cdot} \\ \vdots \\ A_{m\cdot} \end{bmatrix} \text{ and } A = \begin{bmatrix} A_{\cdot 1} & A_{\cdot 2} & A_{\cdot 3} & \ldots & A_{\cdot n} \end{bmatrix}.$$

These are called the row and column block representations of A, respectively

1.1. **Matrix vector Multiplication.** Let $A \in \mathrm{R}^{m\times n}$ and $x \in \mathrm{R}^n$. In terms of its coordinates (or components), we can also write $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ with each $x_j \in \mathrm{R}$. The term x_j is called the jth component of x. For example if

$$x = \begin{bmatrix} 5 \\ -100 \\ 22 \end{bmatrix},$$

then $n = 3,\ x_1 = 5,\ x_2 = -100,\ x_3 = 22$. We define the matrix-vector product Ax by

$$Ax = \begin{bmatrix} A_{1\cdot} \bullet x \\ A_{2\cdot} \bullet x \\ A_{3\cdot} \bullet x \\ \vdots \\ A_{m\cdot} \bullet x \end{bmatrix},$$

where for each $i = 1, 2, \ldots, m$, $A_{i\cdot} \bullet x$ is the dot product of the ith row of A with x and is given by

$$A_{i\cdot} \bullet x = \sum_{j=1}^{n} A_{ij} x_j \ .$$

For example, if

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 & 7 & 3 \\ -2 & 27 & 32 & -100 & 0 & 0 \\ -89 & 0 & 47 & 22 & -21 & 33 \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 2 \\ 3 \end{bmatrix},$$

then

$$Ax = \begin{bmatrix} 24 \\ -29 \\ -32 \end{bmatrix}.$$

Exercise: If

$$C = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} \text{ and } x = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 2 \\ 3 \end{bmatrix},$$

what is Cx?

Note that if $A \in \mathrm{R}^{m\times n}$ and $x \in \mathrm{R}^n$, then Ax is always well defined with $Ax \in \mathrm{R}^m$. In terms of components, the ith component of Ax is given by the dot product of the ith row of A (i.e. $A_{i\cdot}$) and x (i.e. $A_{i\cdot} \bullet x$).

The view of the matrix-vector product described above is the *row-space* perspective, where the term *row-space* will be given a more rigorous definition at a later time. But there is a very different way of viewing the matrix-vector product based on a *column-space* perspective. This view uses the notion of the linear combination of a collection of vectors.

Given k vectors $v^1, v^2, \ldots, v^k \in \mathrm{R}^n$ and k scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathrm{R}$, we can form the vector

$$\alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_k v^k \in \mathrm{R}^n \ .$$

Any vector of this kind is said to be a *linear combination* of the vectors $v^1, v^2, \ldots, v^k$ where the $\alpha_1, \alpha_2, \ldots, \alpha_k$ are called the coefficients in the linear combination. The set of all such vectors formed as linear combinations of $v^1, v^2, \ldots, v^k$ is said to be the *linear span* of $v^1, v^2, \ldots, v^k$ and is denoted

$$\mathrm{Span}\left\{v^1, v^2, \ldots, v^k\right\} := \left\{\alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_k v^k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathrm{R}\right\}.$$

Returning to the matrix-vector product, one has that

$$Ax = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots + A_{1n}x_n \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + \cdots + A_{2n}x_n \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ A_{m1}x_1 + A_{m2}x_2 + A_{m3}x_3 + \cdots + A_{mn}x_n \end{bmatrix} = x_1 A_{\cdot 1} + x_2 A_{\cdot 2} + x_3 A_{\cdot 3} + \cdots + x_n A_{\cdot n},$$

which is a linear combination of the columns of A. That is, we can view the matrix-vector product Ax as taking a linear combination of the columns of A where the coefficients in the linear combination are the coordinates of the vector x.

We now have two fundamentally different ways of viewing the matrix-vector product Ax.

Row-Space view of Ax:

$$Ax = \begin{bmatrix} A_{1\cdot} \bullet x \\ A_{2\cdot} \bullet x \\ A_{3\cdot} \bullet x \\ \vdots \\ A_{m\cdot} \bullet x \end{bmatrix}$$

Column-Space view of Ax:

$$Ax = x_1 A_{\cdot 1} + x_2 A_{\cdot 2} + x_3 A_{\cdot 3} + \cdots + x_n A_{\cdot n} \ .$$

2. Matrix Multiplication

We now build on our notion of a matrix-vector product to define a notion of a matrix-matrix product which we call *matrix multiplication*. Given two matrices $A \in \mathrm{R}^{m\times n}$ and $B \in \mathrm{R}^{n\times k}$ note that each of the columns of B resides in R^n, i.e. $B_{\cdot j} \in \mathrm{R}^n$ $i = 1, 2, \ldots, k$. Therefore, each of the matrix-vector products $AB_{\cdot j}$ is well defined for $j = 1, 2, \ldots, k$. This allows us to define a matrix-matrix product that exploits the block column structure of B by setting

$$AB := \begin{bmatrix} AB_{\cdot 1} & AB_{\cdot 2} & AB_{\cdot 3} & \cdots & AB_{\cdot k} \end{bmatrix}. \tag{1}$$

Note that the jth column of AB is $(AB)_{\cdot j} = AB_{\cdot j} \in \mathrm{R}^m$ and that $AB \in \mathrm{R}^{m\times k}$, i.e.

if $H \in \mathrm{R}^{m\times n}$ and $L \in \mathrm{R}^{n\times k}$, then $HL \in \mathrm{R}^{m\times k}$.

Also note that

if $T \in \mathrm{R}^{s\times t}$ and $M \in \mathrm{R}^{r\times \ell}$, then the matrix product TM is only defined when $t = r$.

For example, if

$$A = \begin{bmatrix} 3 & 2 & -1 & 5 & 7 & 3 \\ -2 & 27 & 32 & -100 & 0 & 0 \\ -89 & 0 & 47 & 22 & -21 & 33 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ -2 & 2 \\ 0 & 3 \\ 0 & 0 \\ 1 & 1 \\ 2 & -1 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} A\begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} & A\begin{bmatrix} 0 \\ -2 \\ 3 \\ 0 \\ 1 \\ -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -58 & 150 \\ -133 & 87 \end{bmatrix}.$$

Exercise: if

$$C = \begin{bmatrix} 3 & -4 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 & 2 & 4 & 3 \\ 0 & -2 & -1 & 4 & 5 \\ 5 & 2 & -4 & 1 & 1 \\ 3 & 0 & 1 & 0 & 0 \end{bmatrix},$$

is CD well defined and if so what is it?

The formula (1) can be used to give further insight into the individual components of the matrix product AB. By the definition of the matrix-vector product we have for each $j = 1, 2, \ldots, k$

$$AB_{\cdot j} = \begin{bmatrix} A_{1\cdot} \bullet B_{\cdot j} \\ A_{2\cdot} \bullet B_{\cdot j} \\ A_{m\cdot} \bullet B_{\cdot j} \end{bmatrix}.$$

Consequently,

$$(AB)_{ij} = A_{i\cdot} \bullet B_{\cdot j} \quad \forall\, i = 1, 2, \ldots m,\ j = 1, 2, \ldots, k.$$

That is, the element of AB in the ith row and jth column, $(AB)_{ij}$, is the dot product of the ith row of A with the jth column of B.

2.1. **Elementary Matrices.** We define the *elementary unit coordinate matrices* in $\mathrm{R}^{m\times n}$ in much the same way as we define the elementary unit coordinate vectors. Given $i \in \{1,2,\dots,m\}$ and $j \in \{1,2,\dots,n\}$, the elementary unit coordinate matrix $E_{ij} \in \mathrm{R}^{m\times n}$ is the matrix whose ij entry is 1 with all other entries taking the value zero. This is a slight abuse of notation since the notation E_{ij} is supposed to represent the ijth entry in the matrix E. To avoid confusion, we reserve the use of the letter E when speaking of matrices to the elementary matrices.

Exercise: (Multiplication of square elementary matrices)*Let $i,k \in \{1,2,\dots,m\}$ and $j,\ell \in \{1,2,\dots,m\}$. Show the following for elementary matrices in $\mathrm{R}^{m\times m}$ first for $m=3$ and then in general.*

(1) $E_{ij}E_{k\ell} = \begin{cases} E_{i\ell} & \text{, if } j=k, \\ 0 & \text{, otherwise.} \end{cases}$

(2) *For any $\alpha \in \mathrm{R}$, if $i \neq j$, then $(I_{m\times m} - \alpha E_{ij})(I_{m\times m} + \alpha E_{ij}) = I_{m\times m}$ so that*

$$(I_{m\times m} + \alpha E_{ij})^{-1} = (I_{m\times m} - \alpha E_{ij}).$$

(3) *For any $\alpha \in \mathrm{R}$ with $\alpha \neq 0$, $(I + (\alpha^{-1}-1)E_{ii})(I + (\alpha-1)E_{ii}) = I$ so that*

$$(I + (\alpha-1)E_{ii})^{-1} = (I + (\alpha^{-1}-1)E_{ii}).$$

Exercise: (Elementary permutation matrices)*Let $i,\ell \in \{1,2,\dots,m\}$ and consider the matrix $P_{ij} \in \mathrm{R}^{m\times m}$ obtained from the identity matrix by interchanging its i and ℓth rows. We call such a matrix an elementary permutation matrix. Again we are abusing notation, but again we reserve the letter P for permutation matrices (and, later, for projection matrices). Show the following are true first for $m=3$ and then in general.*

(1) $P_{i\ell}P_{i\ell} = I_{m\times m}$ *so that* $P_{i\ell}^{-1} = P_{i\ell}$.
(2) $P_{i\ell}^T = P_{i\ell}$.
(3) $P_{i\ell} = I - E_{ii} - E_{\ell\ell} + E_{i\ell} + E_{\ell i}$.

Exercise: (Three elementary row operations as matrix multiplication)*In this exercise we show that the three elementary row operations can be performed by left multiplication by an invertible matrix. Let $A \in \mathrm{R}^{m\times n}$, $\alpha \in \mathrm{R}$ and let $i,\ell \in \{1,2,\dots,m\}$ and $j \in \{1,2,\dots,n\}$. Show that the following results hold first for $m=n=3$ and then in general.*

(1) *(row interchanges) Given $A \in \mathrm{R}^{m\times n}$, the matrix $P_{ij}A$ is the same as the matrix A except with the i and jth rows interchanged.*
(2) *(row multiplication) Given $\alpha \in \mathrm{R}$ with $\alpha \neq 0$, show that the matrix $(I + (\alpha-1)E_{ii})A$ is the same as the matrix A except with the ith row replaced by α times the ith row of A.*
(3) *Show that matrix $E_{ij}A$ is the matrix that contains the jth row of A in its ith row with all other entries equal to zero.*
(4) *(replace a row by itself plus a multiple of another row) Given $\alpha \in \mathrm{R}$ and $i \neq j$, show that the matrix $(I + \alpha E_{ij})A$ is the same as the matrix A except with the ith row replaced by itself plus α times the jth row of A.*

2.2. **Associativity of matrix multiplication.** Note that the definition of matrix multiplication tells us that this operation is associative. That is, if $A \in \mathrm{R}^{m\times n}$, $B \in \mathrm{R}^{n\times k}$, and $C \in \mathrm{R}^{k\times s}$, then $AB \in \mathrm{R}^{m\times k}$ so that $(AB)C$ is well defined and $BC \in \mathrm{R}^{n\times s}$ so that $A(BC)$ is well defined, and, moreover,

$$(AB)C \;=\; \begin{bmatrix}(AB)C_{\cdot 1} & (AB)C_{\cdot 2} & \cdots & (AB)C_{\cdot s}\end{bmatrix} \tag{2}$$

where for each $\ell = 1, 2, \ldots, s$

$$\begin{aligned}(AB)C_{\cdot\ell} &= \begin{bmatrix}AB_{\cdot 1} & AB_{\cdot 2} & AB_{\cdot 3} & \cdots & AB_{\cdot k}\end{bmatrix} C_{\cdot\ell}\\ &= C_{1\ell}AB_{\cdot 1} + C_{2\ell}AB_{\cdot 2} + \cdots + C_{k\ell}AB_{\cdot k}\\ &= A\begin{bmatrix}C_{1\ell}B_{\cdot 1} + C_{2\ell}B_{\cdot 2} + \cdots + C_{k\ell}B_{\cdot k}\end{bmatrix}\\ &= A(BC_{\cdot\ell})\ .\end{aligned}$$

Therefore, we may write (2) as

$$\begin{aligned}(AB)C &= \begin{bmatrix}(AB)C_{\cdot 1} & (AB)C_{\cdot 2} & \cdots & (AB)C_{\cdot s}\end{bmatrix}\\ &= \begin{bmatrix}A(BC_{\cdot 1}) & A(BC_{\cdot 2}) & \ldots & A(BC_{\cdot s})\end{bmatrix}\\ &= A\begin{bmatrix}BC_{\cdot 1} & BC_{\cdot 2} & \ldots & BC_{\cdot s}\end{bmatrix}\\ &= A(BC)\ .\end{aligned}$$

Due to this associativity property, we may dispense with the parentheses and simply write ABC for this triple matrix product. Obviously longer products are possible.

Exercise: *Consider the following matrices:*

$$A = \begin{bmatrix}2 & 3 & 1\\ 1 & 0 & -3\end{bmatrix} \quad B = \begin{bmatrix}4 & -1\\ 0 & -7\end{bmatrix} \quad C = \begin{bmatrix}-2 & 3 & 2\\ 1 & 1 & -3\\ 2 & 1 & 0\end{bmatrix}$$

$$D = \begin{bmatrix}2 & 3\\ 1 & 0\\ 8 & -5\end{bmatrix} \quad F = \begin{bmatrix}2 & 1 & 1 & 2\\ 1 & 0 & -4 & 0\\ 3 & 0 & -2 & 0\\ 5 & 1 & 1 & 1\end{bmatrix} \quad G = \begin{bmatrix}2 & 3 & 1 & -2\\ 1 & 0 & -3 & 0\end{bmatrix}\ .$$

Using these matrices, which pairs can be multiplied together and in what order? Which triples can be multiplied together and in what order (e.g. the triple product BAC is well defined)? Which quadruples can be multiplied together and in what order? Perform all of these multiplications.

3. Block Matrix Multiplication

To illustrate the general idea of block structures consider the following matrix.

$$A = \begin{bmatrix}3 & -4 & 1 & 1 & 0 & 0\\ 0 & 2 & 2 & 0 & 1 & 0\\ 1 & 0 & -1 & 0 & 0 & 1\\ 0 & 0 & 0 & 2 & 1 & 4\\ 0 & 0 & 0 & 1 & 0 & 3\end{bmatrix}\ .$$

Visual inspection tells us that this matrix has structure. But what is it, and how can it be represented? We re-write the the matrix given above *blocking* out some key structures:

$$A = \left[\begin{array}{rrr|rrr} 3 & -4 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right] = \left[\begin{array}{c|c} B & I_{3\times 3} \\ \hline 0_{2\times 3} & C \end{array}\right],$$

where

$$B = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix},$$

$I_{3\times 3}$ is the 3×3 identity matrix, and $0_{2\times 3}$ is the 2×3 zero matrix. Having established this structure for the matrix A, it can now be exploited in various ways. As a simple example, we consider how it can be used in matrix multiplication.

Consider the matrix

$$M = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}.$$

The matrix product AM is well defined since A is 5×6 and M is 6×2. We show how to compute this matrix product using the structure of A. To do this we must first *block decompose* M *conformally with the block decomposition of* A. Another way to say this is that we must give M a block structure that allows us to do block matrix multiplication with the blocks of A. The correct block structure for M is

$$M = \begin{bmatrix} X \\ Y \end{bmatrix},$$

where

$$X = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \end{bmatrix}, \quad \text{and} \quad Y = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix},$$

since then X can multiply $\begin{bmatrix} B \\ 0_{2\times 3} \end{bmatrix}$ and Y can multiply $\begin{bmatrix} I_{3\times 3} \\ C \end{bmatrix}$. This gives

$$\begin{aligned} AM &= \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX+Y \\ CY \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 2 & -11 \\ 2 & 12 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} -2 & 6 \\ 4 & 3 \\ -2 & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 4 & -12 \\ 2 & 9 \\ 0 & 3 \\ 0 & 1 \\ -4 & -1 \end{bmatrix}. \end{aligned}$$

Block structured matrices and their matrix product is a very powerful tool in matrix analysis. Consider the matrices $M \in \mathrm{R}^{n\times m}$ and $T \in \mathrm{R}^{m\times k}$ given by

$$M = \begin{bmatrix} A_{n_1\times m_1} & B_{n_1\times m_2} \\ C_{n_2\times m_1} & D_{n_2\times m_2} \end{bmatrix}$$

and

$$T = \begin{bmatrix} E_{m_1\times k_1} & F_{m_1\times k_2} & G_{m_1\times k_3} \\ H_{m_2\times k_1} & J_{m_2\times k_2} & K_{m_2\times k_3} \end{bmatrix},$$

where $n = n_1 + n_2$, $m = m_1 + m_2$, and $k = k_1 + k_2 + k_3$. The block structures for the matrices M and T are said to be *conformal* with respect to matrix multiplication since

$$MT = \begin{bmatrix} AE+BH & AF+BJ & AG+BK \\ CE+DH & CF+DJ & CG+DK \end{bmatrix}.$$

Similarly, one can conformally block structure matrices with respect to matrix addition (how is this done?).

Exercise: *Consider the the matrix*

$$H = \begin{bmatrix} -2 & 3 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & -3 & 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -7 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 8 & -5 \end{bmatrix}.$$

Does H have a natural block structure that might be useful in performing a matrix-matrix multiply, and if so describe it by giving the blocks? Describe a conformal block decomposition *of the matrix*

$$M = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 1 & -2 \\ -3 & 4 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

that would be useful in performing the matrix product HM. Compute the matrix product HM using this conformal decomposition.

Exercise: *Let $T \in \mathrm{R}^{m\times n}$ with $T \neq 0$ and let I be the $m \times m$ identity matrix. Consider the block structured matrix $A = [\, I \quad T\,]$.*

(i) *If $A \in \mathrm{R}^{k\times s}$, what are k and s?*
(ii) *Construct a non-zero $s \times n$ matrix B such that $AB = 0$.*

The examples given above illustrate how block matrix multiplication works and why it might be useful. One of the most powerful uses of block structures is in understanding and implementing standard *matrix factorizations* or reductions.

4. Gauss-Jordan Elimination Matrices and Reduction to Reduced Echelon Form

In this section, we show that Gaussian-Jordan elimination can be represented as a consequence of left multiplication by a specially designed matrix called a *Gaussian-Jordan elimination matrix.*

Consider the vector $v \in \mathrm{R}^m$ block decomposed as

$$v = \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

where $a \in \mathrm{R}^s$, $\alpha \in \mathrm{R}$, and $b \in \mathrm{R}^t$ with $m = s + 1 + t$. In this vector we refer to the α entry as the *pivot* and assume that $\alpha \neq 0$. We wish to determine a matrix G such that

$$Gv = e_{s+1}$$

where for $j = 1, \ldots, n$, e_j is the unit coordinate vector having a one in the jth position and zeros elsewhere. We claim that the matrix

$$G = \begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix}$$

does the trick. Indeed,

$$Gv = \begin{bmatrix} I_{s\times s} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{t\times t} \end{bmatrix} \begin{pmatrix} a \\ \alpha \\ b \end{pmatrix} = \begin{bmatrix} a - a \\ \alpha^{-1}\alpha \\ -b + b \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_{s+1}. \tag{3}$$

The matrix G is called a *Gaussian-Jordan Elimination Matrix*, or GJEM for short. Note that G is invertible since

$$G^{-1} = \begin{bmatrix} I & a & 0 \\ 0 & \alpha & 0 \\ 0 & b & I \end{bmatrix},$$

Moreover, for any vector of the form $w = \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$ where $x \in \mathrm{R}^s$ $y \in \mathrm{R}^t$, we have

$$Gw = w.$$

The GJEM matrices perform precisely the operations required in order to execute Gauss-Jordan elimination. That is, each elimination step can be realized as left multiplication of the augmented matrix by the appropriate GJEM.

For example, consider the linear system

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 2x_2 & + & 4x_3 & = & 8 \\ 4x_1 & + & 2x_2 & + & 7x_3 & = & 11 \\ 5x_1 & + & 3x_2 & + & 4x_3 & = & 10 \end{array}$$

and its associated augmented matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 2 & 2 & 4 & 8 \\ 4 & 2 & 7 & 11 \\ 5 & 3 & 4 & 10 \end{bmatrix}.$$

The first step of Gauss-Jordan elimination is to transform the first column of this augmented matrix into the first unit coordinate vector. The procedure described in (3) can be employed for this purpose. In this case the pivot is the $(1, 1)$ entry of the augmented matrix and so

$$s = 0,\ a \text{ is void},\ \alpha = 2,\ t = 3,\ \text{and } b = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix},$$

which gives

$$G_1 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -5/2 & 0 & 0 & 1 \end{bmatrix}.$$

Multiplying these two matrices gives

$$G_1 A = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -5/2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 5 \\ 2 & 2 & 4 & 8 \\ 4 & 2 & 7 & 11 \\ 5 & 3 & 4 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 3/2 & 5/2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1/2 & -7/2 & -5/2 \end{bmatrix}.$$

We now repeat this process to transform the second column of this matrix into the second unit coordinate vector. In this case the $(2,2)$ position becomes the pivot so that

$$s=1,\ a=1/2,\ \alpha=1,\ t=2,\ \text{and } b=\begin{bmatrix}0\\1/2\end{bmatrix}$$

yielding

$$G_2=\begin{bmatrix}1 & -1/2 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1/2 & 0 & 1\end{bmatrix}.$$

Again, multiplying these two matrices gives

$$G_2G_1A=\begin{bmatrix}1 & -1/2 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1/2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1/2 & 3/2 & 5/2\\ 0 & 1 & 1 & 3\\ 0 & 0 & 1 & 1\\ 0 & 1/2 & -7/2 & -5/2\end{bmatrix}=\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 1 & 3\\ 0 & 0 & 1 & 1\\ 0 & 0 & -4 & -4\end{bmatrix}.$$

Repeating the process on the third column transforms it into the third unit coordinate vector. In this case the pivot is the $(3,3)$ entry so that

$$s=2,\ a=\begin{bmatrix}1\\1\end{bmatrix},\ \alpha=1,\ t=1,\ \text{and } b=-4$$

yielding

$$G_3=\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 4 & 1\end{bmatrix}.$$

Multiplying these matrices gives

$$G_3G_2G_1A=\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 4 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 1 & 3\\ 0 & 0 & 1 & 1\\ 0 & 0 & -4 & -4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix},$$

which is in reduced echelon form. Therefore the system is consistent and the unique solution is

$$x=\begin{bmatrix}0\\2\\1\end{bmatrix}.$$

Observe that

$$G_3G_2G_1=\begin{bmatrix}3 & -1/2 & -1 & 0\\ 1 & 1 & -1 & 0\\ -2 & 0 & 1 & 0\\ -10 & -1/2 & 4 & 1\end{bmatrix}$$

and that

$$\begin{aligned}(G_3G_2G_1)^{-1} &= G_1^{-1}G_2^{-1}G_3^{-1} \\ &= \begin{bmatrix}2&0&0&0\\2&1&0&0\\4&0&1&0\\5&0&0&1\end{bmatrix}\begin{bmatrix}1&1/2&0&0\\0&1&0&0\\0&0&1&0\\0&1/2&0&1\end{bmatrix}\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&1&0\\0&0&-4&1\end{bmatrix} \\ &= \begin{bmatrix}2&1&3&0\\2&2&4&0\\4&2&7&0\\5&3&4&1\end{bmatrix}.\end{aligned}$$

In particular, reduced Gauss-Jordan form can always be achieved by multiplying the augmented matrix on the left by an invertible matrix which can be written as a product of Gauss-Jordan elimination matrices.

Exercise: *What are the Gauss-Jordan elimination matrices that transform the vector* $\begin{bmatrix}2\\3\\-2\\5\end{bmatrix}$ *in to* e_j *for* $j = 1,2,3,4$, *and what are the inverses of these matrices?*

5. The Four Fundamental Subspaces and Echelon Form

Let $A \in \mathrm{R}^{m\times n}$. We associate with A its four fundamental subspaces:

$$\begin{array}{ll}\mathrm{Ran}\,(A) := \{Ax \mid x \in \mathrm{R}^n\} & \mathrm{Null}\,(A) := \{x \mid Ax = 0\} \\ \mathrm{Ran}\,(A^T) := \left\{A^Ty \mid y \in \mathrm{R}^m\right\} & \mathrm{Null}\,(A^T) := \left\{y \;\middle|\; A^Ty = 0\right\}.\end{array}$$

In the text, it is shown that

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = n \tag{4}$$

and

$$\mathrm{rank}(A^T) + \mathrm{nullity}(A^T) = m,$$

where

$$\begin{array}{ll}\mathrm{rank}(A) := \dim \mathrm{Ran}\,(A) & \mathrm{nullity}(A) := \dim \mathrm{Null}\,(A) \\ \mathrm{rank}(A^T) := \dim \mathrm{Ran}\,(A^T) & \mathrm{nullity}(A^T) := \dim \mathrm{Null}\,(A^T)\end{array} \tag{5}$$

Observe that

$$\begin{aligned}\mathrm{Null}\,(A) &:= \{x \mid Ax = 0\} \\ &= \{x \mid A_{i\cdot} \bullet x = 0,\ i = 1,2,\ldots,m\} \\ &= \{A_{1\cdot}, A_{2\cdot}, \ldots, A_{m\cdot}\}^{\perp} \\ &= \mathrm{Span}\,\{A_{1\cdot}, A_{2\cdot}, \ldots, A_{m\cdot}\}^{\perp} \\ &= \mathrm{Ran}\,(A^T)^{\perp}.\end{aligned}$$

Since for any subspace $S \subset \mathbb{R}^n$, we have $(S^\perp)^\perp = S$, we obtain

(6) $$\mathrm{Null}\,(A)^\perp = \mathrm{Ran}\,(A^T) \text{ and } \mathrm{Null}\,(A^T) = \mathrm{Ran}\,(A)^\perp.$$

The equivalences in (6) are called the *Fundamental Theorem of the Alternative*. By combining this with the rank-plus-nullity equals dimension statement in (4) we find that

$$\mathrm{rank}(A^T) = \dim \mathrm{Ran}\,(A^T) = \dim \mathrm{Null}\,(A)^\perp = n - \mathrm{nullity}(A) = \mathrm{rank}(A).$$

Consequently the row rank of a matrix equals the column rank of a matrix, i.e., the dimensions of the row and column spaces of a matrix are the same!

These observations have consequences for computing bases for the four fundamental subspaces of a matrix using reduced echelon form. Again consider $A \in \mathbb{R}^{m\times n}$ and form the augmented matrix $[A \mid I_m]$. Apply the necessary sequence of Gauss-Jordan eliminations to put this matrix in reduced echelon form. If each column has a non-zero pivot, then the reduced echelon form looks like

(7) $$\left[\begin{array}{cc|cc} I_k & T & R_{11} & R_{12} \\ 0 & 0 & F & I_{(m-k)} \end{array}\right],$$

where $T \in \mathbb{R}^{k\times(n-k)}$, $R \in \mathbb{R}^{k\times k}$, and $F \in \mathbb{R}^{(m-k)\times k}$. This representation tells us that $\mathrm{rank}(A) = k$ and $\mathrm{nullity}(A) = n - k$.

A basis for Ran (A)**:** Observe that (7) implies that

$$GA = \begin{bmatrix} I_k & T \\ 0 & 0 \end{bmatrix},$$

where

$$G = \begin{bmatrix} R_{11} & R_{12} \\ F & I_{(m-k)} \end{bmatrix}$$

is an invertible matrix. In particular,

$$GA_{\cdot j} = \mathrm{e}_j \text{ for } j = 1, 2, \ldots, k.$$

Therefore, the vectors $A_{\cdot j}$, $j = 1, 2, \ldots, k$ are linearly independent (since G is invertible and the vectors e_j for $j = 1, 2, \ldots, k$ are linearly independent). Since $\mathrm{rank}(A) = k$, these vectors necessarily form a basis for $\mathrm{Ran}\,(A)$.

The reduced echelon form (7) provides a second way to obtain a basis for $\mathrm{Ran}\,(A)$. To see this observe that if $b \in \mathrm{Ran}\,(A)$, then we can obtain a solution to $Ax = b$. In particular,

$$GAx = Gb = \begin{bmatrix} R_{11} & R_{12} \\ F & I_{(m-k)} \end{bmatrix} \begin{bmatrix} b^1 \\ b^2 \end{bmatrix},$$

where $b^1 \in \mathbb{R}^k$ and $b^2 \in \mathbb{R}^{(m-k)}$ is a decomposition of b that is conformal to the block structure of G. But then

$$0 = [F \;\, I_{(m-k)}] \begin{bmatrix} b^1 \\ b^2 \end{bmatrix} = Fb^1 + b^2,$$

or equivalently, $b^2 = -Fb^1$. This is a necessary and sufficient condition for b to be an element of $\mathrm{Ran}\,(A)$. Therefore, $b \in \mathrm{Ran}\,(A)$ if and only if there is a vector $b^1 \in \mathrm{R}^k$ such that

$$b = \begin{bmatrix} I_k \\ -F \end{bmatrix} b^1.$$

Since the matrix $\begin{bmatrix} I_k \\ -F \end{bmatrix}$ has $k = \mathrm{rank}(A)$ columns, these columns must form a basis for $\mathrm{Ran}\,(A)$.

Basis for the Null (A)**:** Since G is invertible, we know that $Ax = 0$ if and only if $GAx = 0$. Since $\mathrm{nullity}(A) = n - k$, we need only find $n - k$ linearly independent vectors such that $GAx = 0$, or equivalently, $[I_k\ T]x = 0$. For this, observe that

$$[I\ T] \begin{bmatrix} -T \\ I_{(n-k)} \end{bmatrix} = -T + T = 0,$$

where the matrix $\begin{bmatrix} -T \\ I_{(n-k)} \end{bmatrix}$ has $n - k$ linearly independent columns. Therefore, the columns of this matrix necessarily form a basis for $\mathrm{Null}\,(A)$.

Basis for Ran (A^T)**:** Recall that $\mathrm{A^T} = \mathrm{rank}(A) = k$. Therefore to obtain a basis for A^T we need only obtain k linearly independent vectors in the row space of A. But, by construction, the matrix

$$\begin{bmatrix} I_k & T \\ 0 & 0 \end{bmatrix}$$

is row equivalent to A with the rows of $[I_k\ T]$ necessarily linearly independent. Consequently, the columns of the matrix

$$[I_k\ T]^T = \begin{bmatrix} I_k \\ T^T \end{bmatrix}$$

necessarily form a basis for the range of A^T.

Basis for Null (A^T)**:** By (6), $\mathrm{Null}\,(A^T) = \mathrm{Ran}\,(A)^\perp$. Therefore, to obtain a basis for $\mathrm{Null}\,(A^T)$, we need only find $m - k$ linearly independent vectors in $\mathrm{Ran}\,(A)^\perp$. But we have already seen that the columns of the matrix $\begin{bmatrix} I_k \\ -F \end{bmatrix}$ form a basis for the $\mathrm{Ran}\,(A)$, and

$$[I_k\ \ -F^T] \begin{bmatrix} F^T \\ I_{(m-k)} \end{bmatrix} = F^T - F^T = 0,$$

with the columns of $\begin{bmatrix} F^T \\ I_{(m-k)} \end{bmatrix}$ necessarily linearly independent. Therefore, the columns of this matrix must form a basis for $\mathrm{Null}\,(A^T)$.

Recap: Putting this all together, we find that if $A \in \mathrm{R}^{m\times n}$ is such that $[A\,|\,I]$ has echelon form

$$\left[\begin{array}{cc|cc} I_k & T & R_{11} & R_{12} \\ 0 & 0 & F & I_{(m-k)} \end{array}\right], \tag{8}$$

with $T \in \mathrm{R}^{k\times(n-k)}$, $R \in \mathrm{R}^{k\times k}$, and $F \in \mathrm{R}^{(m-k)\times k}$, then a bases for $\mathrm{Ran}\,(A)$, $\mathrm{Null}\,(A)$, $\mathrm{Ran}\,(A^T)$, and $\mathrm{Null}\,(A^T)$ are given by the columns of the matrices

$$\begin{bmatrix} I_k \\ -F \end{bmatrix}, \quad \begin{bmatrix} -T \\ I_{(n-k)} \end{bmatrix}, \quad \begin{bmatrix} I_k \\ T^{\mathsf{T}} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} F^T \\ I_{(m-k)} \end{bmatrix}, \tag{9}$$

respectively.

Example: *Consider the matrix*

$$A = \begin{bmatrix} 1 & 0 & -4 & 4 \\ 1 & 1 & -1 & 3 \\ 2 & 3 & 1 & 5 \end{bmatrix}.$$

Reducing $[A \,|\, I_3]$ *to reduced echelon form gives*

$$\left[\begin{array}{cccc|ccc} 1 & 0 & -4 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 & 1 \end{array}\right].$$

Using the notation from (8), *we have* $m = 3,\ n = 4,\ k = 2,\ n-k = 2,\ m-k = 1$

$$T = \begin{bmatrix} -4 & 4 \\ 3 & -1 \end{bmatrix}, \quad R_{11} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad R_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad F = [1 \ \ -3].$$

This may be easier to see by conformally partitioning the reduced echelon form of $[A \,|\, I_3]$ *as*

$$\left[\begin{array}{cc|cc} I_k & T & R_{11} & R_{12} \\ 0 & 0 & F & I_{(m-k)} \end{array}\right] = \left[\begin{array}{cc|cc|cc|c} 1 & 0 & -4 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & -3 & 1 \end{array}\right].$$

Then the bases of the four fundamental subspaces are given as the columns of the following matrices:

$$\begin{aligned}
Ran\,(A) \quad &\sim \quad \begin{bmatrix} I_k \\ -F \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} \\
Null\,(A) \quad &\sim \quad \begin{bmatrix} -T \\ I_{(n-k)} \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
Ran\,(A^T) \quad &\sim \quad \begin{bmatrix} I_k \\ T^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -4 & 3 \\ 4 & -1 \end{bmatrix} \\
Null\,(A^T) \quad &\sim \quad \begin{bmatrix} F^T \\ I_{(m-k)} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}.
\end{aligned}$$

Example: *Consider the matrix* $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \end{bmatrix}$. *It has echelon form*

$$G[A \,|\, I] = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 2 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{bmatrix}.$$

Show that the bases of the four fundamental subspaces are given as the columns of the following matrices, and identify which subspace goes with which matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

It may happen that the reduced echelon form of $[A \,|\, I]$ does not take the form given in (8) if a zero pivot occurs. In this case, formulas similar to the ones given in (9) are possible, and can be described in a straightforward manner, but the notational overhead is quite high. Nonetheless, it is helpful to see what is possible, therefore we consider one more reduced echelon structure below. The proof is left to the reader.

Suppose $A \in \mathrm{R}^{m\times n}$ is such that $[A \,|\, I_m]$ has reduced echelon form

$$\left[\begin{array}{cccc|cc} I_{k_1} & T_{11} & 0 & T_{12} & R_{11} & R_{12} \\ 0 & 0 & I_{k_2} & T_{22} & R_{21} & R_{22} \\ 0 & 0 & 0 & 0 & F & I_{m-k} \end{array}\right], \tag{10}$$

where $k := k_1 + k_2$ is the rank of A, $n - k = t_1 + t_2 =: t$ is the nullity of A, $T_{11} \in \mathrm{R}^{k_1\times t_1}$, $T_{12} \in \mathrm{R}^{k_1\times t_2}$, and $T_{22} \in \mathrm{R}^{k_2\times t_2}$. Then the bases of the four fundamental subspaces are given as the columns of the following matrices:

$$\mathrm{Ran}\,(A) \sim \begin{bmatrix} I_k \\ -F \end{bmatrix} \qquad \mathrm{Null}\,(A) \sim \begin{bmatrix} -T_{11} & -T_{12} \\ I_{t_1} & 0 \\ 0 & -T_{22} \\ 0 & I_{t_2} \end{bmatrix}$$

$$\mathrm{Ran}\,(A^T) \sim \begin{bmatrix} I_{k_1} & 0 \\ T_{11}^{\mathsf{T}} & 0 \\ 0 & I_{k_2} \\ T_{12}^{\mathsf{T}} & T_{22}^{\mathsf{T}} \end{bmatrix} \qquad \mathrm{Null}\,(A^T) \sim \begin{bmatrix} F^T \\ I_{(m-k)} \end{bmatrix}.$$

Example: *Consider the matrix*

$$\begin{bmatrix} 1 & 2 & 1 & -2 & 0 \\ 2 & 3 & 4 & -5 & 1 \\ 1 & 5 & -5 & 2 & 0 \\ 2 & 4 & 2 & -4 & 0 \end{bmatrix}.$$

The reduced echelon form for $[A \,|\, I]$ *is the matrix*

$$\left[\begin{array}{ccccc|cccc} 1 & 0 & 5 & 0 & 14 & -31 & 14 & 4 & 0 \\ 0 & 1 & -2 & 0 & -4 & 9 & -4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -7 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right].$$

Show that the bases of the four fundamental subspaces are given as the columns of the following matrices, and identify which subspace goes with which matrix:

$$\begin{bmatrix} -5 & -14 \\ 2 & 4 \\ 1 & 0 \\ 0 & -3 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & -2 & 0 \\ 0 & 0 & 1 \\ 14 & -4 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hint: Identify all of the pieces in the reduced echelon form (10) *from the partitioning*

$$\left[\begin{array}{cccc|cc} I_{k_1} & T_{11} & 0 & T_{12} & R_{11} & R_{12} \\ 0 & 0 & I_{k_2} & T_{22} & R_{21} & R_{22} \\ 0 & 0 & 0 & 0 & F & I_{m-k} \end{array}\right] = \left[\begin{array}{cc|c|c|c|ccc|c} 1 & 0 & 5 & 0 & 14 & -31 & 14 & 4 & 0 \\ 0 & 1 & -2 & 0 & -4 & 9 & -4 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 & 3 & -7 & 3 & 3 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right].$$