Math 308 — Review Material

Linear Systems of Equations from Your Past

Linear systems of equations arise in an enormous variety of real world applications. In addition, our ability to solve very large scale linear systems sometimes involving millions of variables lies at the root of our understanding of the behavior of complex nonlinear phenomenon such as weather. Linear systems lie behind every aspect of the operation of the computer itself, from chip design, to screen layout, to disk space allocation. In this course we will consider a very small number of applications of linear systems and their associated theory. However, it is hoped that this small sampling will give you some idea of the great variety of phenomena that can be described with the aid of this most useful of mathematical theories. After completing Math 308, whole new vistas of mathematical possibility will be open and available to you. This course is the gateway to all upper division courses in mathematics, engineering, and the sciences.

In this handout, I will briefly review a few of the instances in which you were exposed to linear systems in your past courses. The applications are given in a loosely chronological fashion relative to when the associated techniques are introduced into the curriculum.

Partial Fractions Decomposition: (Math 125)

Partial fractions decomposition is a techniques of integration applied to *rational functions*, that is, functions representable as the ratio of polynomials. Given an integrand $f(x) = N(x)/Q(x)$ where N and Q are polynomials, one applies a two step procedure to pre–process f before integration can occur:

Step 1: Use polynomial division to rewrite f as

$$f(x) = M(x) + \frac{P(x)}{Q(x)} ,$$

where M and P are polynomials with the degree of P *strictly* less than the degree of Q.

Step 2: Obtain a partial fractions decomposition of $P(x)/Q(x)$.

After f is pre–processed in this way, integration can occur in a relatively straightforward manner. The aspect of this process that we focus on here is the so called *partial fractions decomposition.* Although a general theory exists, we only consider a specific example to give the flavor of the relationship to linear systems.

Consider the rational function

$$\frac{x^2 - x + 2}{(x^2 + x + 1)(x - 2)} .$$

Observe that the degree of the numerator is 2 and this is strictly less than the degree of the denominator (which is 3). Moreover, the numerator and denominator are *relatively prime*, that is, they have no common zero. Finally, note that the factor $(x^2 + x + 1)$ has only complex roots (these are $\frac{1}{2}(1 + i\sqrt{7})$). In this instance, the theorem on partial fractions decompositions states that there exist coefficients A, B, and C such that

$$\frac{x^2 - x + 2}{(x^2 + x + 1)(x - 2)} = \frac{Ax + B}{(x^2 + x + 1)} + \frac{C}{(x - 2)} \ .$$

In order to obtain the coefficients A, B, and C, one (a) puts the right-hand side of this equation over the common denominator $(x^2+x+1)(x-2)$, (b) multiplies out the numerator, and (c) equates coefficients of the powers of x between the right– and left-hand sides of the equation. In this case, we obtain

$$\frac{x^2 - x + 2}{(x^2 + x + 1)(x - 2)} = \frac{(A + C)x^2 + (-2A + B + C)x + (-2B + C)}{(x^2 + x + 1)(x - 2)} \ .$$

Equating coefficients, we obtain the linear system

$$\begin{array}{rcrcrcr} A & & & + & C & = & 1 \\ -2A & + & B & + & C & = & -1 \\ & - & 2B & + & C & = & 2 \end{array} \ .$$

Problems: Find the partial fractions decomposition for the following functions.

1. $\frac{2x^2+2x+2}{(x^2+1)(x+1)}$
2. $\frac{x^2+2x-1}{(x^2+1)(x+1)^2}$

Polynomial Interpolation: (Pre–Calculus and Beyond)

This example could well have been the first example since it applies to problems that have seen in high school. However, we now take this application to a higher level of abstraction and utility. The most elementary instance of these types of problems is to determine the equation of the line passing through two given points in the plane. This gives rise to the two point formula for the equation of a line. The next most elementary example is to determine the equation of the parabola that passes through three given points in the plane. You have by now worked many problems of this type. These problems are very special cases of the more general class of problems known as *polynomial interpolation* problems.

Again, we consider a special case of polynomial interpolation but at a more general level. Suppose we are given some experimental data. The inputs to the experiment are the distinct values $x_0,\ x_1, \ldots,\ x_n$. Associated with each input value x_i, we measure an output value y_i after the completion of the experiment. After running the same experiment $n+1$ times we acquire $n+1$ pairs of data elements $(x_0, y_0),\ (x_1, y_1), \ldots,\ (x_n, y_n)$ which can be thought of as points in the plane. We now wish to determine a function that associates the inputs with the outputs. That is, we would like to find a function $f(x)$ that *interpolates* the experimental data, i.e., $f(x_i) = y_i$ for $i = 0, 1, \ldots, n$. We can then use this function to gain further insight into the phenomenon that the experiment is attempting to model. In polynomial interpolation with $n+1$ data points, we assume that the function f is a polynomial of degree n:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \ .$$

That is, we wish to determine the coefficients $a_0,\ a_1,\ \ldots, a_n$ so that the resulting polynomial interpolates the data $(x_i, y_i),\ i = 0, 1, \ldots, n$. Specifically, the coefficients must solve the linear system

$$\begin{array}{ccccccccc}
a_0 & + & a_1 x_0 & + & a_2 x_0^2 & + & \ldots & a_n x_0^n & = & y_0 \\
a_0 & + & a_1 x_1 & + & a_2 x_1^2 & + & \ldots & a_n x_1^n & = & y_1 \\
a_0 & + & a_1 x_2 & + & a_2 x_2^2 & + & \ldots & a_n x_2^n & = & y_2 \\
 & & \vdots & & & & \ddots & & & \vdots \\
a_0 & + & a_1 x_n & + & a_2 x_n^2 & + & \ldots & a_n x_n^n & = & y_n
\end{array}$$

Problems:

1. Find the quadratic polynomial that interpolates the 3 points $(1, 0),\ (2, 3)$, and $(-1, 6)$.
2. Find the 3rd degree polynomial that interpolates the 4 points $(0, 1),\ (1, 1),\ (-1, 1)$, and $(2, 7)$.