Linear Programming

Lecture 1: Linear Algebra Review

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- 2 Linear Algebra Review
- Block Structured Matrices
- ④ Gaussian Elimination Matrices
- 5 Gauss-Jordan Elimination (Pivoting)

 $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet 1} & a_{\bullet 2} & \dots & a_{\bullet n} \end{bmatrix} = \begin{bmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{bmatrix}$$
$$A^{T} = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} a_{\bullet}^{T} \\ a_{\bullet}^{T} \\ \vdots \\ a_{\bullet}^{T} \\ \vdots \\ a_{\bullet}^{T} \end{bmatrix}$$

A column space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

 $= x_1 a_{\bullet 1} + x_2 a_{\bullet 2} + \cdots + x_n a_{\bullet n}$

A linear combination of the columns.

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Let $A \in \mathbb{R}^{m \times n}$ (an $m \times n$ matrix having real entries).

Range of A

$$\operatorname{Ran}(A) = \{ y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \text{ such that } y = Ax \}$$

$\operatorname{Ran}(A) =$ the linear span of the columns of A

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Let $v_1, \ldots, v_k \in \mathbb{R}^n$.

• The linear span of v_1, \ldots, v_k :

Span $[v_1, \ldots, v_k] = \{ y \mid y = \xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_k v_k, \xi_1, \ldots, \xi_k \in \mathbb{R} \}$

• The subspace orthogonal to v_1, \ldots, v_k :

$$\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}^{\perp}=\{z\in\mathbb{R}^n\mid z\bullet v_i=0,\ i=1,\ldots,k\}$$

Facts: $\{v_1, \dots, v_k\}^{\perp} = \operatorname{Span} [v_1, \dots, v_k]^{\perp}$ $\operatorname{Span} [v_1, \dots, v_k] = \left[\operatorname{Span} [v_1, \dots, v_k]^{\perp}\right]^{\perp}$

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A row space view of matrix vector multiplication.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1\bullet} \bullet x \\ a_{2\bullet} \bullet x \\ \vdots \\ a_{m\bullet} \bullet x \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_{1i} x_i \\ \sum_{i=1}^n a_{2i} x_i \\ \vdots \\ \sum_{i=1}^n a_{mi} x_i \end{bmatrix}$$

The dot product of *x* with the rows of *A*.

The Null Space of a Matrix

Let $A \in \mathbb{R}^{m \times n}$ (an $m \times n$ matrix having real entries). Null Space of A

$$\operatorname{Nul}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

Nul(A) = subspace orthogonal to the rows of A
=
$$\operatorname{Span}[a_{1\bullet}, a_{2\bullet}, \dots, a_{m\bullet}]^{\perp}$$

= $\operatorname{Ran}(A^{T})^{\perp}$

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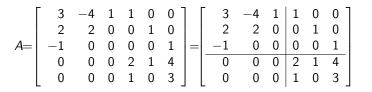
= Ran $(A^T)^{\perp}$

Fundamental Theorem of the Alternative:

$$\operatorname{Nul}(A) = \operatorname{Ran}(A^{\mathsf{T}})^{\perp}$$
 $\operatorname{Ran}(A) = \operatorname{Nul}(A^{\mathsf{T}})^{\perp}$

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$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 1 & | & 1 & 0 & 0 \\ 2 & 2 & 0 & | & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & | & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} B & | I_{3 \times 3} \\ 0_{2 \times 3} & | & C \end{bmatrix}$$

where
$$B = \begin{bmatrix} 3 & -4 & 1 \\ 2 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

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Consider the matrix product AM, where

$$A = \begin{bmatrix} 3 & -4 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}$$

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Can we exploit the structure of A?

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Can we exploit the structure of A?

$$A = \left[\begin{array}{cc} B & I_{3\times3} \\ 0_{2\times3} & C \end{array} \right]$$

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Can we exploit the structure of A?

$$A = \begin{bmatrix} B & I_{3\times3} \\ 0_{2\times3} & C \end{bmatrix} \text{ so take } M = \begin{bmatrix} X \\ Y \end{bmatrix},$$

where $X = \begin{bmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \end{bmatrix}, \text{ and } Y = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}$

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$$AM = \begin{bmatrix} B & I_{3\times 3} \\ 0_{2\times 3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix}$$

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$$AM = \begin{bmatrix} B & I_{3\times3} \\ 0_{2\times3} & C \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} BX + Y \\ CY \end{bmatrix}$$
$$= \begin{bmatrix} \begin{bmatrix} 2 & -11 \\ 2 & 12 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -12 \\ 6 & 15 \\ 1 & -2 \\ 4 & 1 \\ -4 & -1 \end{bmatrix}.$$

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The solution set is either empty, a single point, or an infinite set.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Find all solutions $x \in \mathbb{R}^n$ to the system Ax = b.

The solution set is either empty, a single point, or an infinite set.

If a solution $x_0 \in \mathbb{R}^n$ exists, then the set of solutions is given by

 $x_0 + \operatorname{Nul}(A)$.

We solve the system Ax = b by transforming the augmented matrix

 $[A \mid b]$

into upper echelon form using the three elementary row operations.

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This process is called Gaussian elimination.

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The three elementary row operations.

- Interchange any two rows.
- Multiply any row by a non-zero constant.

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The three elementary row operations.

- Interchange any two rows.
- Multiply any row by a non-zero constant.
- Seplace any row by itself plus a multiple of any *other* row.

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The three elementary row operations.

- Interchange any two rows.
- Multiply any row by a non-zero constant.
- Seplace any row by itself plus a multiple of any other row.

These elementary row operations can be interpreted as multiplying the augmented matrix on the left by a special nonsingular matrix.

Exchange and Permutation Matrices

An exchange matrix is given by permuting any two columns of the identity.

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Multiplying any $4 \times n$ matrix on the left by the exchange matrix

1	0	0	0	٦
0	0	0	1	
0	0	1	0	
0	1	0	0	

will exchange the second and fourth rows of the matrix.

(multiplication of a $m \times 4$ matrix on the right by this exchanges the second and fourth columns.)

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An exchange matrix is given by permuting any two columns of the identity.

Multiplying any $4 \times n$ matrix on the left by the exchange matrix

1	0	0	0	1
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will exchange the second and fourth rows of the matrix.

(multiplication of a $m \times 4$ matrix on the right by this exchanges the second and fourth columns.)

A permutation matrix is obtained by permuting the columns of the identity matrix.

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Notes on Matrix Multiplication

Let $A = [a_{ij}]_{m \times n} \in \mathbb{R}^{m \times n}$.

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Left Multiplication of A:

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When multiplying A on the left by an $m \times m$ matrix M, it is often useful to think of this as an action on the rows of A.

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When multiplying A on the left by an $m \times m$ matrix M, it is often useful to think of this as an action on the rows of A.

For example, left multiplication by a permutation matrix permutes the rows of the matrix.

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When multiplying A on the left by an $m \times m$ matrix M, it is often useful to think of this as an action on the rows of A.

For example, left multiplication by a permutation matrix permutes the rows of the matrix.

However, mechanically, left multiplication corresponds to matrix vector multiplication on the columns.

$$MA = M [a_{\bullet 1} \ a_{\bullet 2} \ \cdots \ a_{\bullet n}] = [Ma_{\bullet 1} \ Ma_{\bullet 2} \ \cdots \ Ma_{\bullet n}]$$

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Right Multiplication of A:

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When multiplying A on the right by an $n \times n$ matrix N, it is often useful to think of this as an action on the columns of A.

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However, mechanically, right multiplication corresponds to left matrix vector multiplication on the rows.

$$AN = \begin{bmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ a_{m\bullet} \end{bmatrix} N = \begin{bmatrix} a_{1\bullet}N \\ a_{2\bullet}N \\ \vdots \\ a_{m\bullet}N \end{bmatrix}$$

The key step in Gaussian elimination is to transform a vector of the form

where $a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$, into one of the form

$$\begin{bmatrix} \mathbf{a} \\ \alpha \\ \mathbf{0} \end{bmatrix}.$$

 $\begin{vmatrix} a \\ \alpha \\ b \end{vmatrix}$,

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 $\begin{bmatrix} a \\ \alpha \\ b \end{bmatrix},$ where $a \in \mathbb{R}^k$, $0 \neq \alpha \in \mathbb{R}$, and $b \in \mathbb{R}^{n-k-1}$, into one of the form $\begin{bmatrix} a \\ \alpha \\ 0 \end{bmatrix}.$

This can be accomplished by left matrix multiplication as follows.

 $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{0} \neq \alpha \in \mathbb{R}$, and $\mathbf{b} \in \mathbb{R}^{n-k-1}$

$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

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$$\begin{bmatrix} I_{k\times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} a \\ \alpha \\ 0 \end{bmatrix}.$$

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Gaussian Elimination Matrices

The matrix

$$G = \begin{bmatrix} I_{k \times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1) \times (n-k-1)} \end{bmatrix}$$

is called a Gaussian elimination matrix.

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is called a Gaussian elimination matrix.

This matrix is invertible with inverse

$$G^{-1} = \begin{bmatrix} I_{k \times k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha^{-1}b & I_{(n-k-1) \times (n-k-1)} \end{bmatrix}$$

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Note that a Gaussian elimination matrix and its inverse are both lower triangular matrices.

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Lower (upper) triangular matrices in $\mathbb{R}^{n \times n}$ are said to form a *sub-algebra* of $\mathbb{R}^{n \times n}$.

A subset S of $\mathbb{R}^{n \times n}$ is said to be a sub-algebra of $\mathbb{R}^{n \times n}$ if

- S is a subspace of $\mathbb{R}^{n \times n}$,
- S is closed wrt matrix multiplication, and
- if $M \in S$ is invertible, then $M^{-1} \in S$.

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Eliminate the first column with a Gaussian elimination matrix. Here k = 0 (so there is no vector *a*), $\alpha = 1$, and $b = (2, -1)^T$. Hence, $-\alpha^{-1}b = (-2, 1)^T$.

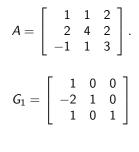
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}.$$
$$G_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

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$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

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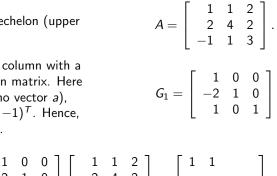
$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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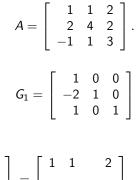
$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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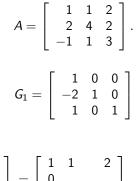
$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

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$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

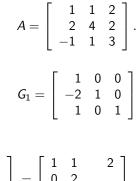
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$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & & \\ & & & \end{bmatrix}$$

Eliminate the first column with a Gaussian elimination matrix. Here k = 0 (so there is no vector *a*), $\alpha = 1$, and $b = (2, -1)^T$. Hence, $-\alpha^{-1}b = (-2, 1)^T$.

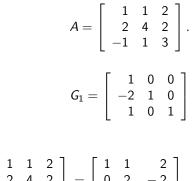


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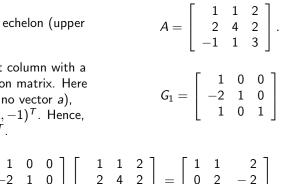
1 0 0 7 5



$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{bmatrix}$$

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$$G_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

In this case k = 1, a = 1, $\alpha = 2$, and b = 2.

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$$\left[\begin{array}{rrrr}1&1&2\\0&2&-2\\0&2&5\end{array}\right] \qquad G_2=\left[\begin{array}{rrrr}1&0&0\\0&1&0\\0&-1&1\end{array}\right]$$

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$$\left[\begin{array}{rrrr}1&1&2\\0&2&-2\\0&2&5\end{array}\right] \qquad G_2=\left[\begin{array}{rrrr}1&0&0\\0&1&0\\0&-1&1\end{array}\right]$$

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$$\left[\begin{array}{rrrr}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{array}\right] \left[\begin{array}{rrrr}1 & 1 & 2\\0 & 2 & -2\\0 & 2 & 5\end{array}\right] = \left[\begin{array}{rrrr}1\\1\end{array}\right]$$

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$$\left[\begin{array}{rrrr}1&1&2\\0&2&-2\\0&2&5\end{array}\right] \qquad G_2=\left[\begin{array}{rrrr}1&0&0\\0&1&0\\0&-1&1\end{array}\right]$$

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In this case k = 1, a = 1, $\alpha = 2$, and b = 2.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 \\ -1 & -2 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & \\ & & & \end{bmatrix}$$

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$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix}$$

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$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

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$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ \end{bmatrix}.$$

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$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0\\ 0 & \alpha^{-1} & 0\\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a\\ \alpha\\ b \end{bmatrix} = \begin{bmatrix} 0\\ 1\\ \end{bmatrix}.$$

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$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0\\ 0 & \alpha^{-1} & 0\\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a\\ \alpha\\ b \end{bmatrix} = \begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}.$$

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$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

What is the inverse of this matrix?

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$$\begin{bmatrix} I_{k\times k} & -\alpha^{-1}a & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & -\alpha^{-1}b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix} \begin{bmatrix} a \\ \alpha \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

What is the inverse of this matrix?

$$\begin{bmatrix} I_{k\times k} & a & 0 \\ 0 & \alpha & 0 \\ 0 & b & I_{(n-k-1)\times(n-k-1)} \end{bmatrix}$$

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