Math 407A: Linear Optimization

Lecture 11: The Dual Simplex Algorithm

Math Dept, University of Washington

The Dual Simplex Algorithm

$$
\begin{array}{lll}
\mathcal{P} & \text{maximize} & -4x_1 - 2x_2 - x_3 \\
& \text{subject to} & -x_1 - x_2 + 2x_3 \le -3 \\
& & -4x_1 - 2x_2 + x_3 \le -4 \\
& & x_1 + x_2 - 4x_3 \le \ \ 2 \\
& & 0 \le x_1, x_2, x_3
\end{array}
$$

The Dual Simplex Algorithm

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & -4x_1 - 2x_2 - x_3 \\ & \text{subject to} & -x_1 - x_2 + 2x_3 \le -3 \\ & & -4x_1 - 2x_2 + x_3 \le -4 \\ & & x_1 + x_2 - 4x_3 \le \ \ 2 \\ & & 0 \le x_1, x_2, x_3 \end{array}$$

$$\begin{array}{lll} \mathcal{D} & \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ & \text{subject to} & -y_1 - 4y_2 + y_3 \ge -4 \\ & & -y_1 - 2y_2 + y_3 \ge -2 \\ & & 2y_1 + y_2 - 4y_3 \ge -1 \\ & & 0 \le y_1, y_2, y_3 \end{array}$$

The Dual Simplex Algorithm

$$\mathcal{P} \quad \begin{array}{lr} \text{maximize} & -4x_1 - 2x_2 - x_3 \\ \text{subject to} & -x_1 - x_2 + 2x_3 \leq -3 \\ & -4x_1 - 2x_2 + x_3 \leq -4 \\ & x_1 + x_2 - 4x_3 \leq \ \ 2 \\ & 0 \leq x_1, x_2, x_3 \end{array}$$

$$\mathcal{D} \quad \begin{array}{lr} \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ \text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\ & -y_1 - 2y_2 + y_3 \geq -2 \\ & 2y_1 + y_2 - 4y_3 \geq -1 \\ & 0 \leq y_1, y_2, y_3 \end{array}$$

$$\begin{array}{|cccccc|c|} -1 & -1 & 2 & 1 & 0 & 0 & -3 \\ -4 & -2 & 1 & 0 & 1 & 0 & -4 \\ 1 & 1 & -4 & 0 & 0 & 1 & 2 \\ \hline -4 & -2 & -1 & 0 & 0 & 0 & 0 \end{array}$$

The Dual Simplex Algorithm

$$
\mathcal{P}\quad
\begin{array}{ll}
\text{maximize} & -4x_1 - 2x_2 - x_3 \\
\text{subject to} & -x_1 - x_2 + 2x_3 \leq -3 \\
 & -4x_1 - 2x_2 + x_3 \leq -4 \\
 & x_1 + x_2 - 4x_3 \leq \ \ 2 \\
 & 0 \leq x_1, x_2, x_3
\end{array}
$$

$$
\mathcal{D}\quad
\begin{array}{ll}
\text{minimize} & -3y_1 - 4y_2 + 2y_3 \\
\text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\
 & -y_1 - 2y_2 + y_3 \geq -2 \\
 & 2y_1 + y_2 - 4y_3 \geq -1 \\
 & 0 \leq y_1, y_2, y_3
\end{array}
$$

-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

Not primal feasible.

Dual feasible!

The dual has feasible origin.

The Dual Simplex Algorithm

The tableau below is said to be *dual feasible* because the objective row coefficients are all non-positive, but it is not *primal feasible*.

-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

The Dual Simplex Algorithm

The tableau below is said to be *dual feasible* because the objective row coefficients are all non-positive, but it is not *primal feasible*.

$$\begin{array}{|cccccc|c|}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0
\end{array}$$

A tableau is optimal if and only if it is both primal feasible **and** dual feasible.

The Dual Simplex Algorithm

The tableau below is said to be *dual feasible* because the objective row coefficients are all non-positive, but it is not *primal feasible*.

-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

A tableau is optimal if and only if it is both primal feasible **and** dual feasible.

Can we design a pivot for this tableau that tries to move it toward primal feasibility while retaining dual feasibility?

The Dual Simplex Algorithm

$\mathcal{D}$

$$\begin{array}{lr} \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ \text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\ & -y_1 - 2y_2 + y_3 \geq -2 \\ & 2y_1 + y_2 - 4y_3 \geq -1 \\ & 0 \leq y_1, y_2, y_3 \end{array}$$

The Dual Simplex Algorithm

$\mathcal{D}$

$$\begin{array}{ll} \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ \text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\ & -y_1 - 2y_2 + y_3 \geq -2 \\ & 2y_1 + y_2 - 4y_3 \geq -1 \\ & 0 \leq y_1, y_2, y_3 \end{array} \qquad \left|\begin{array}{rrrrrr|r} -1 & -1 & 2 & 1 & 0 & 0 & -3 \\ -4 & -2 & 1 & 0 & 1 & 0 & -4 \\ 1 & 1 & -4 & 0 & 0 & 1 & 2 \\ \hline -4 & -2 & -1 & 0 & 0 & 0 & 0 \end{array}\right|$$

The Dual Simplex Algorithm

$\mathcal{D}$

$$\begin{array}{lr}
\text{minimize} & -3y_1 - 4y_2 + 2y_3 \\
\text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\
 & -y_1 - 2y_2 + y_3 \geq -2 \\
 & 2y_1 + y_2 - 4y_3 \geq -1 \\
 & 0 \leq y_1, y_2, y_3
\end{array}$$

$$\begin{array}{|rrrrrr|r|}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0
\end{array}$$

dual
objective
coefficients

The Dual Simplex Algorithm

$\mathcal{D}$

$$\begin{array}{lr} \text{minimize} & -3y_1 - 4y_2 + 2y_3 \\ \text{subject to} & -y_1 - 4y_2 + y_3 \geq -4 \\ & -y_1 - 2y_2 + y_3 \geq -2 \\ & 2y_1 + y_2 - 4y_3 \geq -1 \\ & 0 \leq y_1, y_2, y_3 \end{array}$$

$$\begin{array}{|cccccc|c|}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{} \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{y_1} & \multicolumn{1}{c}{y_2} & \multicolumn{1}{c}{y_3} & \multicolumn{1}{c}{}
\end{array}$$

dual
objective
coefficients

Dual variables

The Dual Simplex Algorithm

$\mathcal{D}$

$$\begin{array}{lrl} \text{minimize} & -3y_1 - 4y_2 + 2y_3 & \\ \text{subject to} & -y_1 - 4y_2 + y_3 & \geq -4 \\ & -y_1 - 2y_2 + y_3 & \geq -2 \\ & 2y_1 + y_2 - 4y_3 & \geq -1 \\ & 0 \leq y_1, y_2, y_3 & \end{array}$$

$$\begin{array}{|rrrrrr|r|} -1 & -1 & 2 & 1 & 0 & 0 & -3 \\ -4 & -2 & 1 & 0 & 1 & 0 & -4 \\ 1 & 1 & -4 & 0 & 0 & 1 & 2 \\ \hline -4 & -2 & -1 & 0 & 0 & 0 & 0 \\ \multicolumn{1}{r}{} & & & \uparrow & \uparrow & \uparrow & \multicolumn{1}{r}{} \\ \multicolumn{1}{r}{} & & & y_1 & y_2 & y_3 & \multicolumn{1}{r}{} \end{array}$$

dual
objective
coefficients

Dual variables

Increasing y_1 decreases the value of the dual objective.

The Dual Simplex Algorithm

Increasing y_1 means we pivot on row 1.

$$\left|\begin{array}{rrrrrr|r}
-1 & -1 & 2 & 1 & 0 & 0 & -3 \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 \\
1 & 1 & -4 & 0 & 0 & 1 & 2 \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0
\end{array}\right|$$

The Dual Simplex Algorithm

Increasing y_1 means we pivot on row 1.

$$\begin{array}{|rrrrrr|r|l}
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}$$

The Dual Simplex Algorithm

Increasing y_1 means we pivot on row 1.

$$\begin{array}{|rrrrrr|r|l}
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}$$

By how much can we increase the value of y_1?

The Dual Simplex Algorithm

Increasing y_1 means we pivot on row 1.

$$\begin{array}{|rrrrrr|r|l}
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}$$

By how much can we increase the value of y_1?

$$\begin{array}{r|}
-y_1 - 4y_2 + y_3 \geq -4 \\
-y_1 - 2y_2 + y_3 \geq -2 \\
2y_1 + y_2 - 4y_3 \geq -1
\end{array}$$

The Dual Simplex Algorithm

Increasing y_1 means we pivot on row 1.

$$\begin{array}{|rrrrrr|r|l}
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}$$

By how much can we increase the value of y_1?

$$\begin{array}{r|l}
 & \text{ratio} \\
-y_1 - 4y_2 + y_3 \geq -4 & 4/1 \\
-y_1 - 2y_2 + y_3 \geq -2 & 2/1 \\
2y_1 + y_2 - 4y_3 \geq -1 &
\end{array}$$

The Dual Simplex Algorithm

Increasing y_1 means we pivot on row 1.

$$
\begin{array}{|rrrrrr|r|l}
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 & \\
\multicolumn{1}{c}{4/1} & \multicolumn{1}{c}{2/1} & & & & & & \quad\text{ratios}
\end{array}
$$

By how much can we increase the value of y_1?

$$
\begin{array}{r|l}
 & \text{ratio} \\
-y_1 - 4y_2 + y_3 \geq -4 & 4/1 \\
-y_1 - 2y_2 + y_3 \geq -2 & 2/1 \\
2y_1 + y_2 - 4y_3 \geq -1 &
\end{array}
$$

The Dual Simplex Algorithm

Increasing y_1 means we pivot on row 1.

$$
\begin{array}{|rrrrrr|r|l}
-1 & \boxed{\text{-1}} & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 & \\
\multicolumn{1}{r}{4/1} & \multicolumn{1}{r}{2/1} & & & & & & \text{ratios}
\end{array}
$$

By how much can we increase the value of y_1?

$$
\begin{array}{r|l}
 & \text{ratio} \\
-y_1 - 4y_2 + y_3 \geq -4 & 4/1 \\
-y_1 - 2y_2 + y_3 \geq -2 & 2/1 \\
2y_1 + y_2 - 4y_3 \geq -1 &
\end{array}
$$

The Dual Simplex Algorithm

-1	-1	2	1	0	0	-3
-4	-2	1	0	1	0	-4
1	1	-4	0	0	1	2
-4	-2	-1	0	0	0	0

The Dual Simplex Algorithm

$$\begin{array}{|cccccc|c|l}
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}$$

The Dual Simplex Algorithm

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\begin{array}{|rrrrrr|r|l}
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}
$$

Any row having a negative rhs is a candidate pivot row.

The Dual Simplex Algorithm

$$\begin{array}{|cccccc|c|l}
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}$$

Any row having a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

The Dual Simplex Algorithm

$$\begin{array}{|rrrrrr|r|l}
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}$$

Any row having a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

The pivot column is given by the smallest ratio.

The Dual Simplex Algorithm

$$\begin{array}{|rrrrrr|r|l}
 & \begin{array}{c}\text{pivot}\\ \text{column}\\ \downarrow\end{array} & & & & & & \\
-1 & -1 & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}$$

Any row having a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

The pivot column is given by the smallest ratio.

The Dual Simplex Algorithm

$$\begin{array}{|cccccc|c|l}
 & \begin{array}{c}\text{pivot}\\ \text{column}\\ \downarrow\end{array} & & & & & & \\
-1 & \boxed{-1} & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row}\\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 &
\end{array}$$

Any row having a negative rhs is a candidate pivot row.

Form the ratios with the negative entries in pivot row.

The pivot column is given by the smallest ratio.

The Dual Simplex Algorithm

$$\begin{array}{|cccccc|c|l}
-1 & \boxed{-1} & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 & \\
\hline
\end{array}$$

$$\begin{array}{|cccccc|c|}
1 & 1 & -2 & -1 & 0 & 0 & 3 \\
-2 & 0 & -3 & -2 & 1 & 0 & 2 \\
0 & 0 & -2 & 1 & 0 & 1 & -1 \\
\hline
-2 & 0 & -5 & -2 & 0 & 0 & 6
\end{array}$$

The Dual Simplex Algorithm

$$\begin{array}{|cccccc|c|l}
-1 & \boxed{-1} & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 & \\
\hline
\end{array}$$

$$\begin{array}{|cccccc|c|l}
1 & 1 & -2 & -1 & 0 & 0 & 3 & \\
-2 & 0 & -3 & -2 & 1 & 0 & 2 & \\
0 & 0 & -2 & 1 & 0 & 1 & -1 & \leftarrow \text{pivot row} \\
\hline
-2 & 0 & -5 & -2 & 0 & 0 & 6 &
\end{array}$$

The Dual Simplex Algorithm

$$\begin{array}{|cccccc|c|l}
-1 & \boxed{-1} & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -4 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 & \\
\hline
\end{array}$$

$$\begin{array}{|cccccc|c|l}
1 & 1 & -2 & -1 & 0 & 0 & 3 & \\
-2 & 0 & -3 & -2 & 1 & 0 & 2 & \\
0 & 0 & \boxed{-2} & 1 & 0 & 1 & -1 & \leftarrow \text{pivot row} \\
\hline
-2 & 0 & -5 & -2 & 0 & 0 & 6 &
\end{array}$$

The Dual Simplex Algorithm

1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	$\boxed{-2}$	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	
1	1	0	-2	0	-1	4	
-2	0	0	-7/2	1	-3/2	7/2	
0	0	1	-1/2	0	-1/2	1/2	
-2	0	0	-9/2	0	-5/2	17/2	

The Dual Simplex Algorithm

1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	$\boxed{-2}$	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	
1	1	0	-2	0	-1	4	
-2	0	0	-7/2	1	-3/2	7/2	
0	0	1	-1/2	0	-1/2	1/2	
-2	0	0	-9/2	0	-5/2	17/2	optimal

The Dual Simplex Algorithm

$$
\begin{array}{|rrrrrr|r|l}
1 & 1 & 0 & -2 & 0 & -1 & 4 & \\
-2 & 0 & 0 & -7/2 & 1 & -3/2 & 7/2 & \\
0 & 0 & 1 & -1/2 & 0 & -1/2 & 1/2 & \\
\hline
-2 & 0 & 0 & -9/2 & 0 & -5/2 & 17/2 & \text{optimal}
\end{array}
$$

The Dual Simplex Algorithm

$$\begin{array}{|cccccc|c|l}
1 & 1 & 0 & -2 & 0 & -1 & 4 & \\
-2 & 0 & 0 & -7/2 & 1 & -3/2 & 7/2 & \\
0 & 0 & 1 & -1/2 & 0 & -1/2 & 1/2 & \\
\hline
-2 & 0 & 0 & -9/2 & 0 & -5/2 & 17/2 & \text{optimal}
\end{array}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1/2 \end{pmatrix}$$

The Dual Simplex Algorithm

$$\begin{array}{|rrrrrr|r|l}
1 & 1 & 0 & -2 & 0 & -1 & 4 & \\
-2 & 0 & 0 & -7/2 & 1 & -3/2 & 7/2 & \\
0 & 0 & 1 & -1/2 & 0 & -1/2 & 1/2 & \\
\hline
-2 & 0 & 0 & -9/2 & 0 & -5/2 & 17/2 & \text{optimal}
\end{array}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 0 \\ 5/2 \end{pmatrix},$$

The Dual Simplex Algorithm

$$\begin{array}{|cccccc|c|l}
1 & 1 & 0 & -2 & 0 & -1 & 4 & \\
-2 & 0 & 0 & -7/2 & 1 & -3/2 & 7/2 & \\
0 & 0 & 1 & -1/2 & 0 & -1/2 & 1/2 & \\
\hline
-2 & 0 & 0 & -9/2 & 0 & -5/2 & 17/2 & \text{optimal}
\end{array}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 0 \\ 5/2 \end{pmatrix},$$

Optimal value $= -17/2$.

The Dual Simplex Algorithm

Apply the dual simplex algorithm to the following problem.

$$\begin{array}{lll} \mathcal{P} & \text{maximize} & -4x_1 - 2x_2 - x_3 \\ & \text{subject to} & -x_1 - x_2 + 2x_3 \leq -3 \\ & & -4x_1 - 2x_2 + x_3 \leq -4 \\ & & x_1 + x_2 - x_3 \leq 2 \\ & & 0 \leq x_1, x_2, x_3 \ . \end{array}$$

The Dual Simplex Algorithm

$$\begin{array}{|cccccc|c|l}
-1 & \boxed{-1} & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -1 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 & \\
\hline
 & & & & & & &
\end{array}$$

The Dual Simplex Algorithm

-1	$\boxed{-1}$	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	
-2	0	-5	-2	0	0	6	

The Dual Simplex Algorithm

-1	$\boxed{-1}$	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

The Dual Simplex Algorithm

$$
\begin{array}{|cccccc|c|l}
-1 & \boxed{-1} & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -1 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 & \\
\hline
1 & 1 & -2 & -1 & 0 & 0 & 3 & \\
-2 & 0 & -3 & -2 & 1 & 0 & 2 & \\
0 & 0 & 1 & 1 & 0 & 1 & -1 & \leftarrow \text{pivot row} \\
\hline
-2 & 0 & -5 & -2 & 0 & 0 & 6 &
\end{array}
$$

No negative entry in the pivot row!
What does this mean?

The Dual Simplex Algorithm

-1	$\boxed{-1}$	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

No negative entry in the pivot row!
What does this mean?

The dual problem is unbounded.

The Dual Simplex Algorithm

-1	$\boxed{-1}$	2	1	0	0	-3	← pivot row
-4	-2	1	0	1	0	-4	
1	1	-1	0	0	1	2	
-4	-2	-1	0	0	0	0	
1	1	-2	-1	0	0	3	
-2	0	-3	-2	1	0	2	
0	0	1	1	0	1	-1	← pivot row
-2	0	-5	-2	0	0	6	

No negative entry in the pivot row!
What does this mean?

The dual problem is unbounded.
What can you say about the primal problem?

The Dual Simplex Algorithm

$$\begin{array}{|rrrrrr|r|l}
-1 & \boxed{-1} & 2 & 1 & 0 & 0 & -3 & \leftarrow \text{pivot row} \\
-4 & -2 & 1 & 0 & 1 & 0 & -4 & \\
1 & 1 & -1 & 0 & 0 & 1 & 2 & \\
\hline
-4 & -2 & -1 & 0 & 0 & 0 & 0 & \\
\hline
1 & 1 & -2 & -1 & 0 & 0 & 3 & \\
-2 & 0 & -3 & -2 & 1 & 0 & 2 & \\
0 & 0 & 1 & 1 & 0 & 1 & -1 & \leftarrow \text{pivot row} \\
\hline
-2 & 0 & -5 & -2 & 0 & 0 & 6 &
\end{array}$$

No negative entry in the pivot row!
What does this mean?

The dual problem is unbounded.
What can you say about the primal problem?
The primal is necessarily infeasible by the Weak Duality Theorem.

The Dual Simplex Algorithm

Solve the following LP using the dual simplex algorithm.

$$\begin{array}{lrcrcrcr}
\text{maximize} & -4x_1 & - & 3x_2 & - & 2x_3 & & \\
\text{subject to} & x_1 & & & - & x_3 & \leq & -1 \\
 & -x_1 & - & x_2 & & & \leq & -2 \\
 & x_1 & - & x_2 & - & 2x_3 & \leq & 0 \\
 & 0 & \leq & x_1, & x_2, & x_3 & &
\end{array}$$