MATH 408 FINAL EXAM Due 10:00am, March 16, 2020

NAME (Print Clearly): ______________________________

Student Number: ______________________________

Problem	Pts	Score
1	20	
2	30	
3	40	
4	30	
5	40	
6	40	
Total	200	

RULES: Please read!

(1) You are expected to work on this problem by yourself with no outside assistance.
(2) Show all of your work and follow the directions provided. Partial credit will be given for partial solutions.
(3) This is an open book and open notes take-home exam. It is expected that the Final Exam Guide, homework solutions, and the course notes will be your primary sources for constructing solutions. The final exam and guide have been specifically designed to complement each other.
(4) You are requested to use the internet to only access the course website.

Good Luck!

(1) (20 points) Compute and classify all critical points of the function $f(x_1, x_2) := (x_1+x_2)^2-8(x_1+x_2)$.

(2) (30 points) Show that the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \exp(\frac{1}{2} \|x\|_2^2)$ is convex.

(3) Consider the problem $\min \{ x_1^2 - x_2 \mid x_1^2 + x_2 \leq 0 \}$.

(a) (5 points) Graph the constraint region $\{(x_1,\ x_2) \mid x_1^2 + x_2 \leq 0\}$.

(b) (20 points) Compute a KKT pair for this problem.

(c) (5 points) Compute the tangent cone to the constraint region at the solution to this problem.

(d) (10 points) Show that the second-order sufficiency condition for this problem is satisfied at the KKT pair computed above.

(4) (30 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and let $\mathsf{e} \in \mathbb{R}^n$ denote the vector of all ones. Show that if $\bar{x}$ is a local solution to the problem $\min \{f(x) \; | 0 \le x \le \mathsf{e}\}$, then

$$\frac{\partial f}{\partial x_i}(\bar{x}) \ge 0 \quad \text{if } \bar{x}_i = 0,$$
$$\frac{\partial f}{\partial x_i}(\bar{x}) = 0 \quad \text{if } 0 < \bar{x}_i < 1,$$
$$\frac{\partial f}{\partial x_i}(\bar{x}) \le 0 \quad \text{if } \bar{x}_i = 1.$$

Hint: KKT conditions and $\{x \; | 0 \le x \le \mathsf{e}\} = \{x \; | 0 \le x_i \le 1, \; i = 1, 2, \ldots, n\}$.

(5) Consider the problem

$$\text{minimize } (x_1 + x_2)^2 - 8(x_1 + x_2)$$
$$\text{subject to } \; x_1^2 \le 2x_2 \quad \text{and} \quad 2x_1 + 2x_2 \le 4,$$

and note that this objective function occurs in problem 1.

(a) (10 points) Graph the constraint region and compare it to the graph of the set of critical points in problem 1. After thinking about the geometry of the setting, guess that one of the two dual variables takes the value zero.

(b) (30 points) Describe the set of all KKT pairs for this problem.

(6) (40 points) Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n \setminus \{0\}$, and $\gamma \in \mathbb{R}$. Show that the Lagrangian dual for the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|Ax-b\|_2^2 \\ \text{subject to} & c^T x = \gamma \quad \text{and} \quad 0 \le x, \end{array}$$

is the problem

$$\begin{array}{ll} \text{maximize} & -\frac{1}{2}\|y+b\|_2^2 - \lambda\gamma + \frac{1}{2}\|b\|_2^2 \\ \text{subject to} & 0 \le A^T y + \lambda c\,, \end{array}$$

where the maximization occurs over the dual variables $y \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$.

Step 1: Rewrite the problem by introducing a new variable w that simplifies the objective. Don't forget to write the definition of the new variable as one of the constraints.

Step 2: Write the Lagrangian.

Step 3: Write the condition $0 = \nabla_{(x,w)} L$.

Step 4: Use this condition to eliminate the primal variables from L and obtain the dual objective as a function of the dual variables only.

Step 5: Clean up the dual problem a bit so that it corresponds to the one give above.