FAILURE OF THE HASSE PRINCIPLE FOR CHÂTELET SURFACES IN CHARACTERISTIC 2

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Abstract. Given any global field k of characteristic 2, we construct a Châtelet surface over k that fails to satisfy the Hasse principle. This failure is due to a Brauer-Manin obstruction. This construction extends a result of Poonen to characteristic 2, thereby showing that the étale-Brauer obstruction is insufficient to explain all failures of the Hasse principle over a global field of any characteristic.

1. Introduction

Poonen recently showed that, for a global field k of characteristic different from 2, the étale-Brauer obstruction is insufficient to explain failures of the Hasse principle [6]. This result relied on the existence of a Châtelet surface over k that violates the Hasse principle [5, Prop 5.1 and §11]. Poonen's construction fails in characteristic 2 due to the inseparability of $y^2 - az^2$.

Classically, Châtelet surfaces have only been studied over fields of characteristic different from 2. In this paper, we define Châtelet surfaces over fields of characteristic 2 and obtain a result analogous to [5, Prop 5.1].

Theorem 1.1. *Let k be any global field of characteristic 2. There exists a Châtelet surface X over k that violates the Hasse principle.*

The only assumption on characteristic in [6] is in using [5, Prop 5.1] (all other arguments go through exactly as stated after replacing any polynomial of the form $by^2 + az^2$ by its Artin-Schreier analogue, $by^2 + byz + az^2$). Therefore, Theorem 1.1 extends the main result of [6] to global fields of characteristic 2, thereby showing that the étale-Brauer obstruction is insufficient to explain all failures of the Hasse principle over a global field of any characteristic.

The proof of Theorem 1.1 is constructive. The difficulty in the proof lies in finding suitable equations so that the Brauer set is easy to compute and empty.

2. Background

2.1. **Brauer-Manin obstructions.** The counterexamples to the Hasse principle referred to in Theorem 1.1 are all explained by the Brauer-Manin obstruction, which we recall here [1, Thm. 1]. Let k be a global field and let $\mathbb{A}_k$ be the adèle ring of k. Recall that for a projective variety X, we have the equality $X(\mathbb{A}_k) = \prod_v X(k_v)$, where v runs over all nontrivial places of k. The Brauer group of X, denoted $\operatorname{Br} X$, is the group of equivalence classes of Azumaya algebras on X. Let inv_v denote the morphism from class field theory $\operatorname{Br}\mathbb{Q}_v \to \mathbb{Q}/\mathbb{Z}$ [7, XIII.3, Prop. 6]. For any $\mathcal{A} \in \operatorname{Br} X$, $P \in X(S)$, let $\operatorname{ev}_{\mathcal{A}}(P)$ be the image of $\mathcal{A}$ under the map $P^*\colon \operatorname{Br} X \to \operatorname{Br} S$, which is induced by $P\colon S \to X$. Define

This research was supported by the Mentored Research Award from University of California, Berkeley.

$$X(\mathbb{A}_k)^{\mathrm{Br}} := \left\{ (P_v)_v \in X(\mathbb{A}_k) \colon \sum_v \mathrm{inv}_v \left(\mathrm{ev}_{\mathcal{A}}(P_v)\right) = 0 \text{ for all } \mathcal{A} \in \operatorname{Br} X \right\}$$

By class field theory we have

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k).$$

Thus, if $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, then X has no k-points. We say there is a Brauer-Manin obstruction to the Hasse principle if $X(\mathbb{A}_k) \neq \emptyset$ but $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. See [8, §5.2] for more details.

2.2. **Châtelet surfaces in characteristic 2.** A conic bundle X over $\mathbb{P}^1$ is the zero-locus of a nowhere-vanishing global section s of $\operatorname{Sym}^2(\mathcal{E})$ in $\mathbb{P}\mathcal{E}$, for some rank 3 vector sheaf $\mathcal{E}$ on $\mathbb{P}^1$. Consider the special case where $\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2)$ and $s = s_1 - s_2$ where s_1 is a global section of $\operatorname{Sym}^2(\mathcal{O} \oplus \mathcal{O})$ and s_2 is a global section of $\mathcal{O}(2)^{\otimes 2} = \mathcal{O}(4)$. Take $a \in k^\times$ and $P(x)$ a separable polynomial over k of degree 3 or 4. If $s_1 = y^2 + yz + az^2$ and $s_2 = w^4 P(x/w)$, then X contains the affine variety defined by $y^2 + yz + az^2 = P(x)$ as an open subset. In this case we say X is the Châtelet surface defined by

$$y^2 + yz + az^2 = P(x).$$

By the same basic argument used in [5, Lemma 3.1], we can show that X is smooth. See [5, §3 and §5] for the construction of a Châtelet surface in the case where the characteristic is different from 2.

3. Proof of Theorem 1.1

Let k denote a global field of characteristic 2. Let $\mathbb{F}$ denote its constant field and let n denote the order of $\mathbb{F}^\times$. Fix a prime $\mathfrak{p}$ of k of odd degree (recall that the degree of a prime $\mathfrak{p}$ is the degree of the field extension $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}$ over $\mathbb{F}$), and let $S = \{\mathfrak{p}\}$. Let $\mathcal{O}_{k,S}$ denote the ring of S-integers. Let $\gamma \in \mathbb{F}$ be such that $T^2 + T + \gamma$ is irreducible in $\mathbb{F}[T]$. By the Chebotarev density theorem [4, Thm 13.4, p. 545] applied to the compositum of $k[T]/(T^2 + T + \gamma)$ and the Hilbert class field of k, we can find an element $b \in \mathcal{O}_{k,S}$ that generates a prime of odd degree. Similarly, by applying the Chebotarev density theorem to the compositum of $k[T]/(T^2 + T + \gamma)$ and the ray class field of modulus b^2, we can find an element $a \in \mathcal{O}_{k,S}$ that generates a prime of even degree and that is congruent to $\gamma \pmod{b^2 \mathcal{O}_{k,S}}$. These conditions imply that $v_{\mathfrak{p}}(a)$ is even and negative and that $v_{\mathfrak{p}}(b)$ is odd and negative.

Define

$$\begin{aligned} f(x) &= a^{-4n} b x^2 + x + a b^{-1}, \\ g(x) &= a^{-8n} b^2 x^2 + a^{-4n} b x + a^{1-4n} + \gamma. \end{aligned}$$

Note that $g(x) = a^{-4n} b f(x) + \gamma$. Let X be the Châtelet surface given by

$$y^2 + yz + \gamma z^2 = f(x) g(x). \tag{$*$}$$

In Lemma 3.1 we show $X(\mathbb{A}_k) \neq \emptyset$, and in Lemma 3.3 we show $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. Together, these show that X has a Brauer-Manin obstruction to the Hasse principle.

Lemma 3.1. *The Châtelet surface X has a k_v-point for every place v.*

Proof. Suppose that $v = v_a$. Since a generates a prime of even degree, the left-hand side of $(*)$ factors into two distinct linear factors in $k_v[y,z]$. Therefore, we can change variables so that $(*)$ becomes $w_1w_2 = f(x)g(x)$ and hence there is a solution over k_v.

Now suppose that $v \neq v_a$. Since $y^2 + yz + \gamma z^2$ is a norm form for an unramified extension of k_v for all v, in order to prove the existence of a k_v-point, it suffices to find an $x \in k_v$ such that the valuation of the right-hand side of $(*)$ is even.

Suppose further that $v \neq v_{\mathfrak{p}}, v_b$. Choose x such that $v(x) = -1$. Then the right-hand side of $(*)$ has valuation -4 so there exists a k_v-point.

Suppose that $v = v_{\mathfrak{p}}$. Let π be a uniformizer for v and take $x = \pi a^2/b$. Then

$$f(x) \quad = \quad b^{-1}a^{4-4n}\pi^2 + a^2b^{-1}\pi + ab^{-1}.$$

Since a has negative even valuation and $n \geq 1$, we have $v(f(x)) = v(a^2b^{-1}\pi)$ which is even. Now let us consider

$$g(x) \quad = \quad a^{4-8n}\pi^2 + a^{2-4n}\pi + a^{1-4n} + \gamma.$$

By the same conditions mentioned above, all terms except for γ have positive valuation. Therefore $v(g(x)) = 0$.

Finally suppose that $v = v_b$. Take $x = \frac{1}{b} + 1$. Then

$$f(x) = \frac{1}{b}\left(a^{-4n} + a + 1 + b + a^{-4n}b^2\right).$$

Note that by the conditions imposed on a, $(a^{-4n} + a + 1 + b + a^{-4n}b^2) \equiv \gamma + b \pmod{b^2\mathcal{O}_{k,S}}$. Thus $v(f(x)) = -1$. Now consider

$$g(x) = a^{-8n} + a^{-8n}b^2 + a^{-4n} + a^{-4n}b + a^{1-4n} + \gamma$$

modulo $b^2\mathcal{O}_{k,S}$. By the conditions imposed on a, we have

$$g(x) \equiv 1 + 1 + b + \gamma + \gamma \equiv b \pmod{b^2\mathcal{O}_{k,S}}.$$

Thus $v(g(x)) = 1$, so $v\left(f(x)g(x)\right)$ is even. □

Let $L = k[T]/(T^2 + T + \gamma)$ and let $\mathcal{A}$ denote the class of the cyclic algebra $(L/k, f(x))_2$ in $\operatorname{Br} k(X)$(see [2, §2.5] for a detailed definition). Using the defining equation of the surface, we can show that $(L/k, g(x))_2$ is also a representative for $\mathcal{A}$. Since $g(x) + a^{-4n}bf(x)$ is a v-adic unit, $g(x)$ and $f(x)$ have no common zeroes. Since $\mathcal{A}$ is the class of a cyclic algebra of order 2, the algebra $(L/k, f(x)/x^2)_2$ is another representative for $\mathcal{A}$. Note that for any point P of X, there exists an open neighborhood U containing P such that either $f(x)$, $g(x)$, or $f(x)/x^2$ is a nowhere vanishing regular function on U. Therefore, $\mathcal{A}$ is an element of $\operatorname{Br} X$.

To show that $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, we use the continuity of the map $\mathrm{ev}_{\mathcal{A}}$. While the result is well-known, it is difficult to find in the literature so we give a proof for reader's convenience.

Lemma 3.2. *Let k_v be a local field and let V be a smooth projective scheme over k_v. For any $[\mathcal{B}] \in \operatorname{Br} V$,*

$$\mathrm{ev}_{\mathcal{B}} : V(k_v) \to \operatorname{Br} k_v$$

is locally constant.

Proof. To prove continuity, it suffices to show that $\mathrm{ev}_{\mathcal{B}}^{-1}(\mathcal{B}')$ is open for any $\mathcal{B}'$ in the image of $\mathrm{ev}_{\mathcal{B}}$. By replacing $[\mathcal{B}]$ with $[\mathcal{B}] - [\mathrm{ev}_{\mathcal{B}}(x)]$, we reduce to showing that $\mathrm{ev}_{\mathcal{B}}^{-1}(0)$ is open.

Fix a representative $\mathcal{B}$ of the element $[\mathcal{B}] \in \operatorname{Br} V$. Let n^2 denote the rank of $\mathcal{B}$ and let $f_{\mathcal{B}} \colon Y_{\mathcal{B}} \to V$ be the PGL_n-torsor associated to $\mathcal{B}$. Then we observe that the set $\mathrm{ev}_{\mathcal{B}}^{-1}(0)$ is equal to $f_{\mathcal{B}}(Y_{\mathcal{B}}(k_v)) \subset V(k_v)$. This set is open by the implicit function theorem [3, Thm. 2.2.1]. $\square$

Lemma 3.3. *Let $P_v \in X(k_v)$. Then*

$$\mathrm{inv}_v(\mathrm{ev}_{\mathcal{A}}(P_v)) = \begin{cases} 1/2 & \text{if } v = v_b, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.

Proof. The surface X contains an open affine subset that can be identified with

$$V(y^2 + yz + az^2 - P(x)) \subseteq \mathbb{A}^3.$$

Let X_0 denote this open subset. Since $\mathrm{ev}_{\mathcal{A}}$ is continuous by Lemma 3.2 and inv_v is an isomorphism onto its image, it suffices to prove that inv_v takes the desired value on the v-adically dense subset $X_0(k_v) \subset X(k_v)$.

Since L/k is an unramified extension for all places v, evaluating the invariant map reduces to computing the parity of the valuation of $f(x)$ or $g(x)$.

Suppose that $v \neq v_a, v_b, v_{\mathfrak{p}}$. If $v(x_0) < 0$, then by the strong triangle inequality, $v(f(x_0)) = v(x_0^2)$. Now suppose that $v(x_0) \geq 0$. Then both $f(x_0)$ and $g(x_0)$ are v-adic integers, but since $g(x) - a^{-4n}bf(x) = \gamma$ either $f(x_0)$ or $g(x_0)$ is a v-adic unit. Thus, for all $P_v \in X_0(k_v)$, $\mathrm{inv}_v(\mathcal{A}(P_v)) = 0$.

Suppose that $v = v_a$. Since a generates a prime of even degree, $T^2 + T + \gamma$ splits in k_a. Therefore, $(L/k, h)$ is trivial for any $h \in k_a(V)^{\times}$ and so $\mathrm{inv}_v(\mathcal{A}(P_v)) = 0$ for all $P_v \in X_0(k_v)$.

Suppose that $v = v_{\mathfrak{p}}$. We will use the representative $(L/k, g(x))$ of $\mathcal{A}$. If $v(x_0) < v(a^{4n}b^{-1})$ then the quadratic term of $g(x_0)$ has even valuation and dominates the other terms. If $v(x_0) > v(a^{4n}b^{-1})$ then the constant term of $g(x_0)$ has even valuation and dominates the other terms. Now assume that $x_0 = a^{4n}b^{-1}u$, where u is a v-adic unit. Then we have

$$g(x_0) = u^2 + u + \gamma + a^{1-4n}.$$

Since γ was chosen such that $T^2+T+\gamma$ is irreducible in $\mathbb{F}[T]$ and $\mathfrak{p}$ is a prime of odd degree, $T^2 + T + \gamma$ is irreducible in $\mathbb{F}_{\mathfrak{p}}[T]$. Thus, for any v-adic unit u, $u^2 + u + \gamma \not\equiv 0 \pmod{\mathfrak{p}}$. Since $a^{1-4n} \equiv 0 \mod \mathfrak{p}$, this shows $g(x_0)$ is a v-adic unit. Hence $\mathrm{inv}_v(\mathcal{A}(P_v)) = 0$ for all $P_v \in X_0(k_v)$.

Finally suppose that $v = v_b$. We will use the representative $(L/k, f(x))$ of $\mathcal{A}$. If $v(x_0) < -1$ then the quadratic term has odd valuation and dominates the other terms in $f(x_0)$. If $v(x_0) > -1$ then the constant term has odd valuation and dominates the other terms in $f(x_0)$. Now assume $x_0 = b^{-1}u$ where u is any v-adic unit. Then we have

$$f(x_0) = \frac{1}{b}\left(a^{-4n}u^2 + u + a\right).$$

It suffices to show that $a^{-4n}u^2 + u + a \not\equiv 0 \pmod{b\mathcal{O}_{k,S}}$. Since $a \equiv \gamma \pmod{b\mathcal{O}_{k,S}}$, we have

$$a^{-4n}u^2 + u + a \equiv \overline{u}^2 + \overline{u} + \gamma.$$

Using the same argument as in the previous case, we see that $a^{-4n}u^2+u+a \not\equiv 0 \pmod{b\mathcal{O}_{k,s}}$ and thus $v(g(x_0)) = -1$. Therefore $\mathrm{inv}_v(\mathcal{A}(P_v)) = \frac{1}{2}$ for all $P_v \in X_0(k_v)$. $\square$

Acknowledgements

I thank my advisor, Bjorn Poonen, for suggesting the problem and for many helpful conversations. I thank Olivier Wittenberg for a sketch of the proof of Lemma 3.2 and thank Daniel Erman and the referees for their comments.

References

[1] Y. I. Manin, *Le groupe de Brauer-Grothendieck en géométrie diophantienne.* Actes du Congrès International des Mathématiciens (Nice, 1970), Gauthier-Villars, Paris, (1971), 401–411.

[2] Philippe Gille, Tamás Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics, vol. 101, Cambridge University Press, Cambridge, 2006.

[3] Jun-ichi Igusa, *An introduction to the theory of local zeta functions*, AMS/IP Studies in Advanced Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2000.

[4] Jürgen Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Translated from the 1992 German original and with a note by Norbert Schappacher; With a foreword by G. Harder, Springer-Verlag, Berlin, 1999.

[5] Bjorn Poonen, *Existence of rational points on smooth projective varieties*, J. Eur. Math. Soc. (JEMS), **11** (2009), no. 3, 529–543.

[6] Bjorn Poonen, *Insufficiency of the Brauer-Manin obstruction applied to étale covers*, Ann. of Math. (2), **171** (2010), no. 3, 2157–2169.

[7] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Translated from the French by Marvin Jay Greenberg, Springer-Verlag, New York, 1979.

[8] Alexei Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, Cambridge, 2001.

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