Linear-to-linear Example

There are basically three types of problems that require the determination of a linear-to-linear function. The three types are based on the kind of information given about the function. The three types are:

1. You know three points the the graph of the function passes through;
2. You know one of the function's asymptotes and two points the graph passes through;
3. You know both asymptotes and one point the graph passes through.

Notice that in all cases you know **three pieces of information**. Since a linear-to-linear function is determined by three parameters, this is exactly the amount of information needed to determine the function.

The worst case, in terms of the amount of algebra you need to do, is the first case. Let's look at an example of this sort.

Example: Find the linear-to-linear rational function $f(x)$ such that $f(10) = 20$, $f(20) = 32$ and $f(25) = 36$.

Since $f(x)$ is a linear-to-linear rational function, we know

$$f(x) = \frac{ax+b}{x+c}$$

for constants a, b, and c. We need to find a, b and c.

We know three things.

First, $f(10) = 20$ so

$$f(10) = \frac{10a+b}{10+c} = 20$$

which we can rewrite as

$$10a + b = 200 + 20c \tag{1}$$

Second, $f(20) = 32$ so

$$f(20) = \frac{20a+b}{20+c} = 32$$

which we can rewrite as

$$20a + b = 640 + 32c \tag{2}$$

Third, $f(25) = 36$ so

$$f(25) = \frac{25a + b}{25 + c} = 36$$

which we can rewrite as

$$25a + b = 900 + 36c \tag{3}$$

These three numbered equations are enough algebraic material to solve for a, b, and c. Here is one way to do that.

Subtract equation (1) from equation (2) to get

$$10a = 440 + 12c \tag{4}$$

and subtract equation (2) from equation (3) to get

$$5a = 260 + 4c \tag{5}$$

Note that we've eliminated b. Now multiply (5) by 2 to get

$$10a = 520 + 8c$$

Subtract equation (4) from this to get

$$0 = 80 - 4c$$

which easily give us $c = 20$.

Plugging this value into equation (4), we can find $a = 68$, and then we can find $b = -80$.

Thus,

$$f(x) = \frac{68x - 80}{x + 20}.$$

We can check that we've done the algebra correctly be evaluating $f(x)$ at $x = 10$, $x = 20$ and $x = 25$. If we get $f(10) = 20$, $f(20) = 32$ and $f(25) = 36$, then we'll know our work is correct.