Probabilistic gauge for periodic and almost periodic potentials

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1 The periodic case

Let X denote Brownian motion on the line and q a continuous nonconstant periodic function with least positive period 1. Put

$$u(x) = u(x;q) = E_x e_q(\sigma),\ x > 0, \tag{1}$$

where

$$e_q(\sigma) = \exp(\int_0^\sigma q(X_t)dt),\ \sigma = \sigma_0 = \min\{t : X_t = 0\}.$$

A simple calculation shows that

$$u(x+1) = u(1)E_{x+1}e_q(\sigma_1) = u(1)u(x).$$

The function u, if finite, satisfies

$$\tfrac{1}{2}u'' + qu = 0 \text{ on } (0,\infty) \text{ with } u(0+) = 1. \tag{2}$$

See [1], Chapter IX. The function $v(x) = u(x)\int_0^x u(z)^{-2}dz,\ x \geq 0$, also satisfies the differential equation. From this one sees that

$$\int_0^\infty u(z)^{-2}dz = \infty, \tag{3}$$

for if not, then $w = u - v/k$, where k is the value of the integral on the left in (3), would be a positive solution to the differential equation with initial value 1 smaller than the minimal solution u. Since u and v span the solutions it also follows that u is the only non-negative solution to (2) which also satisfies (3). Chung calls u the *gauge* for $\{(0,\infty);q\}$. (For additional pertinent information see also [2], Chapter XI, Theorem 6.4.) These facts about u lead to the following Floquet type result.

Theorem 1 *If u is finite then $u(x)=e^{-cx}r(x), x>0$, where r is a strictly positive continuous periodic function (with period 1) and the number c is nonnegative. A necessary (but not sufficient) condition for u to be finite is that $\int_0^1 q(x)dx<0$.*

The last assertion follows upon integrating the identity $2q=-p^2-p'$ where $p=u'/u$. For an example with $\int_0^1 q(x)dx<0$ but $u\equiv\infty$, take $q(x)=-1+360\cos(2\pi x)$. Then, for $0<x<\frac{1}{6}$, we have

$$u(x)\geq E_x[e_q(\sigma_0);\sigma_0<\sigma_{\frac{1}{6}}]\geq E_x[\exp(179\sigma_0);\sigma_0<\sigma_{1/6}]=\infty,$$

which implies $u(x)=\infty$ for all $x>0$. The case $c=0$ in Theorem 1 can occur; $u(x)=\exp(\sin x)$ is the gauge of $\{(0,\infty);q\}$ where $q=-\frac{1}{2}u''/u$.

2 The almost periodic case

We have in Theorem 1 another instance of a class of functions, periodic, defined on a fixed domain, $(0,\infty)$, such that for each q in the class either the gauge u is identically infinite or else it exists and is bounded. This pleasant dichotomy disappears when we replace the class of periodic functions by the larger and more interesting class of almost periodic functions.

A continuous function f on R is almost periodic if to each $\varepsilon>0$ there is a number $\ell(\varepsilon)>0$ such that every interval of length $\ell(\varepsilon)$ contains a number γ (an ε-almost period) for which $|f(x+\gamma)-f(x)|<\varepsilon$ for all x.

A basic result is this: A function f is almost periodic if and only if for every $\varepsilon>0$ there is a trigonometric "polynomial" $T_\varepsilon(x)=\sum_{k=1}^{r} b_k e^{i\mu_k x}$ such that $\|f-T_\varepsilon\|\equiv\sup\{|f(x)-T_\varepsilon(x)|;-\infty<x<\infty\}<\varepsilon$. A series of the form $\sum_{k=1}^{\infty} A_k e^{i\mu_k x}$ which converges uniformly in x on R defines an almost periodic function.

If f is an almost periodic function, then f is bounded, uniformly continuous, and the mean, $M\{f\}\equiv\lim_{T\to\infty}(1/T)\int_0^T f(x)dx$ exists. Moreover, every $f^*\in \mathrm{Hull}\{f\}$, the closure in the uniform topology of the set of translates of f, is also almost periodic and has the same mean.

The next theorem provides the basic Chung-gauge result for general almost periodic potentials.

Theorem 2 *Let q be a non-constant almost periodic function and for any constant a let $u_a(x) = u(x;\, q-a) = E^x e_{q-a}(\sigma_0)$ be the gauge corresponding to $q(\cdot) - a$. Then there exists an number $a_0 > M\{q\}$ such that for $a > a_0$, $u_a(x)$, as a function of x, decays exponentially at ∞, but $u_a(\cdot) \equiv \infty$ for $a < a_0$. For $a = a_0$, $u_{a_0}(x)$ is finite but may be unbounded. Moreover, the same conclusions with the same number a_0 hold for every q^* in the* Hull$\{q\}$. *If the derivative dq/dx exists and is also almost periodic, then there exists at least one $q^* \in$* Hull$\{q\}$ *such that the gauge $u(x; q^* - a_0)$ is bounded.*

We will not prove Theorem 2 here. One may easily construct a proof using some results in [4]; in particular Theorems 1, 2, 10 (and its Corollaries 1 and 2), and Theorem 16.

We call the number $a_0 = a_0(q)$ of Theorem 2 the *gauge number of q*.

Example. This example shows that the gauge corresponding to an almost periodic potential need not be bounded. Put

$$g(x) = \sum_{n=1}^{\infty} \sin(\gamma_n x), \qquad \gamma_n = \frac{1}{5 \cdot 7 \ldots (4n+1)}$$

The series defining g converges uniformly on bounded intervals but does not define an almost periodic function for, as we will see, $\limsup_{x \to +\infty} g(x) = +\infty$ and $\liminf_{x \to +\infty} g(x) = -\infty$. However g can be differentiated term by term with the result

$$g'(x) = \sum_{n=1}^{\infty} \gamma_n \cos(\gamma_n x),$$

and this is an almost periodic function since this series converges uniformly on R. Likewise g'' is almost periodic. Let

$$q = -\tfrac{1}{2}(g')^2 - \tfrac{1}{2}g''.$$

Then q is almost periodic (with mean $M\{q\} = -\frac{1}{4}\sum_{n=1}^{\infty} \gamma_n^2$) and the positive continuous function $u = e^g$ satisfies (2). Furthermore u is the smallest positive solution to (2) because, as we will show, (3) holds for u. These facts guarantee the finiteness of the expectation in (1) and establish that the function u defined here coincides with the gauge $u(x;q)$ as defined in (1). The unboundedness above of g implies the same for u. Note, however, that $\liminf_{x \to \infty} u(x) = 0$. Also, from Theorem 2, it follows that the gauge number, $a_0(q)$, equals 0.

It remains to verify the unboundedness assertions. Let

$$x_j = 5 \cdot 9 \dots (4j+1)(\pi/2) = \pi/2 \mod 2\pi.$$

Then $\sin(\gamma_n x_j) = +1$ for $n = 1, 2, \dots, j$, and $\sin(\gamma_n x_j) = 0(1/n), 0(1/n^2)$ for $n = j+1$, $n \geq j+2$, respectively. Hence $g(x_j) = j + 0(1/j) \to +\infty$ as $j \to \infty$. Next let

$$y_j = (4j+3)x_j = 3\pi/2 \mod 2\pi.$$

Then $g(y_j) = -j + 0(1) \to -\infty$. Similarly, on the interval $y_j - 1 \leq x \leq y_j + 1$, $g(x) \leq -j\cos(\pi/10) + 2 \leq -1/2$, for $j \geq 3$, and therefore

$$\int_0^\infty u(x)^{-2} dx = \int_0^\infty e^{-2g(x)} dx \geq \sum_j 2e = \infty.$$

3 A conjecture for almost periodic polynomial potentials

Fix $n+1$ numbers $b_0, b_1, \dots, b_n$, with $b_j \neq 0$, $j \geq 1$, and n real numbers $\mu_1, \mu_2, \dots, \mu_n$. We assume that the later set of numbers is linearly independent over the rationals. For $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n) \in [0, 2\pi)^n$ let us put

$$q(x; \boldsymbol{\theta}) = b_0 + \sum_{k=1}^{n} b_k \cos(\mu_k x + \theta_k), \tag{4}$$

We also put $q_0(x, \boldsymbol{\theta})$ for the case in which $b_0 = 0$. These functions are almost periodic with $M\{q(\cdot\,; \boldsymbol{\theta})\} = b_0$.

For any fixed $\boldsymbol{\theta^0}$ the linear independence of the μs and Kronecker's Theorem ([3], Theorem 444) implies that for every positive ϵ, there exist arbitrarily large numbers t and integers $r_1, r_2, \dots, r_n$ such that

$$|t\mu_k + 2r_k\pi - \theta_k^0| < \epsilon$$

for $k = 1, 2, \dots, n$. Hence there is a sequence t_j, tending to infinity with j, such that $\cos(\mu_k(x + t_j)) \to \cos(\mu_k x + \theta_k^0)$ uniformly for x in $(-\infty, \infty)$, $k = 1, 2, \dots, n$, which implies $q(x + t_j\,; 0) \to q(x\,; \boldsymbol{\theta}^0)$ uniformly in x. We can apply the same sort of argument to any function $q(\cdot\,; \boldsymbol{\theta})$ of (4). The upshot is that that the hull of any function of the form (4) does not depend on $\boldsymbol{\theta}$ and coincides exactly with the collection of functions (4) that one gets as $\boldsymbol{\theta}$ varies over $[0, 2\pi)^n$ (but with the same numbers $\mu_1, \ldots, \mu_n, b_0, \ldots, b_n$, of course).

Let a_0 denote the gauge number of the function $q_0(x, \boldsymbol{\theta})$. This number does not depend on $\boldsymbol{\theta}$. From Theorem 2 it follows that the "Gauge Dichotomy" holds for $\{q, (0, \infty)\}$ for any q of the form (4) *provided* $b_0 \neq -a_0$. For $b_0 = -a_0$, the gauge $u(x; q(\cdot\,; \theta)$ is at least finite for all $\boldsymbol{\theta} \in [0, 2\pi)^n$ and is bounded in x for at least one $\boldsymbol{\theta}$.

It is tempting to conjecture that the gauge is a bounded function of x for every $\boldsymbol{\theta}$ in particular for the case $b_0 = -a_0$.

References

[1] Chung, K.L. and Zhao, Z. *From Brownian Motion to Schrödinger's Equation.* Springer-Verlag, New York. 1995.

[2] Hartman, P. *Ordinary Differential Equations.* Birkhauser, Boston. 1982.

[3] Hardy, G.H., and Wright, E. M. *An Introduction to the Theory of Numbers, 5th ed.* Clarendon Press, Oxford. 1979.

[4] Markus, L. and Moore, R.A. Oscillation and disconjugacy for linear differential equations with almost periodic coefficients. *Acta Math.* **96** (1956). 99-123.