Assignment 1. Due Friday, Jan. 16.

Reading: Course Notes, through p. 19.
Coddington and Levinson, Ch. 1, secs. 1–6.
Skim through: Birkhoff and Rota, Ch. 1, secs. 1–6, 9–10; Ch. 2, secs. 1–2, 4; Ch. 3, secs. 1–5, Ch. 6, secs. 1–3.

1. **(a)** Let $f : \mathbf{R} \times \mathbf{C}^n \to \mathbf{C}$ be continuous. Suppose $x : \mathbf{R} \to \mathbf{C}$ is a solution of the nth-order equation
$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}); \tag{1}$$
i.e., for each $t \in \mathbf{R}$, $x^{(n)}(t)$ exists and $x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t))$. Show that $x \in C^n(\mathbf{R})$.

   (b) Define $F : \mathbf{R} \times \mathbf{C}^n \to \mathbf{C}^n$ by $F(t, y) = [y_2, \ldots, y_n, f(t, y_1, \ldots, y_n)]^T$ (so F is continuous). Suppose $y : \mathbf{R} \to \mathbf{C}^n$ is a solution of the first-order system
$$y' = F(t, y); \tag{2}$$
i.e., for each $t \in \mathbf{R}$, $y'(t)$ exists and $y'(t) = F(t, y(t))$. Show that $y \in C^1(\mathbf{R})$, and moreover for $1 \le j \le n$, $y_j \in C^{n-j+1}(\mathbf{R})$.

   (c) Show that if $x \in C^n(\mathbf{R})$ is a solution of (1), then $y = [x, x', \ldots, x^{(n-1)}]^T$ is a C^1 solution of (2). Moreover, if x satisfies the initial conditions $x^{(k)}(t_0) = x_0^k$ $(0 \le k \le n-1)$, then y satisfies the initial conditions $y(t_0) = [x_0^0, \ldots, x_0^{n-1}]^T$.

   (d) Show that if y is a C^1 solution of (2), then $x = y_1$ is a C^n solution of (1). Moreover, if y satisfies the initial conditions $y(t_0) = y_0$, then x satisfies the initial conditions $x^{(k)}(t_0) = (y_0)_{k+1}$ $(0 \le k \le n-1)$.

   (e) Show that the first-order system corresponding to the linear nth-order equation $x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_n(t)x = b(t)$ is of the form $y' = A(t)y + B(t)$ where $A(t) \in \mathbf{C}^{n\times n}$ and $B(t) \in \mathbf{C}^n$, and identify $A(t)$ and $B(t)$.

2. For each of the following IVP's, compute the Picard iterates and identify the solution to which they converge.

   (a) $x' = tx$, $\quad x(0) = 1$. (x is a scalar function of t.)

   (b) $x' = Ax$, $\quad x(0) = x_0$. ($A \in \mathbf{C}^{n\times n}$ is a constant matrix and $x : \mathbf{R} \to \mathbf{C}^n$.)

3. Let $f \in C$ $(n = 1)$ on the rectangle $0 \le t \le a$, $|x| \le b$, where $a, b > 0$, and assume that $f(t, x_1) \le f(t, x_2)$ if $x_1 \le x_2$, and $f(t, 0) \ge 0$ for $0 \le t \le a$. Prove that the successive approximations in the Picard iteration:
$$x_{k+1}(t) = \int_0^t f(s, x_k(s))\, ds, \quad k = 0, 1, 2, \ldots$$
converge to a solution of $x' = f(t, x)$, $x(0) = 0$, on $[0, \min\{a,\ b/M\}]$, where $M = \max|f|$ on the rectangle.

4. For each pair of positive constants M and T, define a norm $\|\cdot\|_{M,T}$ on $C([0,T])$ by

$$\|x\|_{M,T} = \sup_{0\le t\le T} |e^{-tM}x(t)|.$$

(a) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is uniformly Lipschitz continuous and that $T > 0$ is given. Show that the mapping

$$\Phi : x \to x_0 + \int_0^t f(x(s))\, ds$$

is a contraction on $C([0,T])$ in the $\|\cdot\|_{M,T}$ norm as long as M is large enough.

(b) Apply this to show the existence of a unique solution $x \in C^1([0,T])$ to the initial value problem

$$x' = f(x), \quad x(0) = x_0$$

on a finite interval $[0,T]$ of any length $T > 0$.

5. Consider the initial-value problem

$$\begin{aligned} y'(t) &= f(y(t/2)) \text{ for } t \ge 0, \\ y(0) &= y_0, \end{aligned}$$

where $f : \mathbf{R} \to \mathbf{R}$ is uniformly Lipschitz continuous with Lipschitz constant L and y_0 is a given constant. This is a "differential delay" equation: notice that y on the right-hand side above is evaluated at $t/2$, not at t, so this is not a standard ODE. Prove that there exists a unique solution of this "differential delay" initial-value problem in $C^1[0,\infty)$.