Assignment 4. Due Fri., Feb. 6.

Reading: Course Notes, through p. 44.

1. This problem illustrates the use of the linearization equation to approximate solutions. On a particle of unit mass near the surface of the earth, gravity imparts a constant downward acceleration of $g \approx 9.8\text{m}/sec^2$, so Newton's equation is $x'' = -g$, where x is the height of the particle above the earth's surface. Let $x_0(t)$ denote the solution corresponding to a particle dropped from rest at height $h > 0$. Suppose now that we want to include the effect of air resistance, which we model as a small frictional force proportional to the square of the velocity. Then the equation becomes $x'' = -g + \epsilon(x')^2$, where $\epsilon > 0$ is small.

   (a) Find and solve the linearized equation in ϵ about the solution $x_0(t)$.

   (b) Use your result from (a) to estimate, to first order in ϵ, the amount of time longer that it takes the particle to reach the ground as a result of air resistance. For $h = 100\text{m}$, calculate the free-fall time and also the estimated increase due to air resistance if $\epsilon = 0.001$.

2. (2001 prelim, problem 1) Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) Determine the Jordan form of A. Explicitly compute S and J in the decomposition $A = SJS^{-1}$.

   (b) Explicitly write down the solution to the initial value problem:

$$y'(t) = Ay(t), \quad y(0) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

3. Let $A \in \mathbf{C}^{n \times n}$ and let $\lambda \in \mathbf{C}$ be an eigenvalue of A of algebraic multiplicity 2 but geometric multiplicity 1. If $\mathbf{v} \in \mathbf{C}^n$ is an eigenvector associated with λ, then $e^{\lambda t}\mathbf{v}$ is a solution of $x' = Ax$. We want to find a second linearly independent solution corresponding to eigenvalue λ.

   (a) Try another solution of the form $x(t) = e^{\lambda t}(t\mathbf{y} + \mathbf{z})$, with $\mathbf{y}, \mathbf{z} \in \mathbf{C}^n$. Derive conditions on $\mathbf{y}$ and $\mathbf{z}$ in order that $x(t)$ solve the system. Show that one can always find $\mathbf{y}$ and $\mathbf{z}$ to obtain a solution of this form.

(b) Find a basis for the solution set of the system $x' = Ax$ with

$$A = \begin{pmatrix} 8 & 4 \\ -9 & -4 \end{pmatrix}.$$

(c) Find an explicit similarity transformation to put A (in part(b)) into Jordan form.

(d) (**Not** to turn in.) Think about how you could generalize this: How can you explicitly compute the Jordan form of a matrix? How can you explicitly write down a basis for the solution set of $x' = Ax$ in general?

4. Let $A : \mathbf{R} \to \mathbf{C}^{n\times n}$ be continuous and suppose that

$$\liminf_{t\to\infty} \operatorname{Re}\left(\int_0^t \operatorname{tr}(A(s))\,ds\right) > -\infty.$$

Suppose that $\Phi(t)$ is a fundamental matrix for the system $x' = A(t)x$, and suppose that Φ is uniformly bounded on $[0,\infty)$ (in some norm).

(a) Show that Φ^{-1} is uniformly bounded on $[0,\infty)$.

(b) Show that no nontrivial solution to $x' = A(t)x$ can satisfy $x(t) \to 0$ as $t \to +\infty$.

5. Let $A \in \mathbf{C}^{n\times n}$.

(a) Prove that $\det(e^A) = e^{\operatorname{tr}(A)}$ by considering the Wronskian of the normalized fundamental matrix of the system $x' = Ax$.

(b) Give a second proof using the spectral mapping theorem.

6. A first-order autonomous system of differential equations for unknowns $(x_1, \ldots, x_n, p_1, \ldots, p_n) \equiv (x,p) \in \mathbf{R}^{2n}$ is said to be a *Hamiltonian system* if there is a function $H(x,p)$ on $\mathbf{R}^{2n}$ (real-valued, usually assumed to be C^2, called the *Hamiltonian function*) so that the system is of the form

$$x_i' = \frac{\partial H}{\partial p_i}(x,p), \quad p_i' = -\frac{\partial H}{\partial x_i}(x,p), \quad 1 \le i \le n.$$

In mechanics, (x_i, p_i) represents the position and momentum of the ith particle in a system, and $H(x,p)$ is the total energy of the system.

(a) Suppose H is of the form

$$H(x,p) = \frac{1}{2}\sum_{i=1}^{n} \frac{p_i^2}{m_i} + V(x_1, \ldots, x_n),$$

where the first term represents the kinetic energy and the second term the potential energy, assumed to depend only on the positions of the particles, and $m_i > 0$ is the mass of the ith particle. Show that the system above is equivalent to Newton's equations, $F_i = m_i a_i$ for the n particles, where $F_i = -\partial V/\partial x_i$ is the force on the ith particle and $a_i = x_i''$ is its acceleration.

(b) Suppose $(x(t), p(t))$ is a solution of a Hamiltonian system. Show that $H(x(t), p(t))$ is independent of t.

(c) Suppose $f : \mathbf{R}^m \to \mathbf{R}^m$ is a C^1 vector field. Let $x(t, y)$ denote the solution of the IVP $x' = f(x)$, $x(0) = y$ for $y \in \mathbf{R}^m$. The vector field f is said to define a *volume-preserving flow* if for each open set $\mathcal{U} \in \mathbf{R}^m$ and each $t \in \mathbf{R}$, $\text{vol}\left(\{x(t, y) : y \in \mathcal{U}\}\right) = \text{vol}(\mathcal{U})$. It can be shown (don't do it here) that f defines a volume-conserving flow if and only if $\text{div} f \equiv \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_i} = 0$. Prove Liouville's Theorem: The flow defined by any Hamiltonian system is volume-conserving in $\mathbf{R}^{2n}$. (You may assume for simplicity that all solutions exist for all time.)