Assignment 5. Due Friday, Feb. 20.

Reading: Course Notes, chapter on numerical solution of IVP's

1. Consider the implicit one-step method

$$x_{i+1} = x_i + \frac{h}{2}[f(t_i, x_i) + f(t_{i+1}, x_{i+1})],$$

Show that the local truncation error,

$$x(t+h) - \left(x(t) + \frac{h}{2}[f(t, x(t)) + f(t+h, x(t+h))]\right),$$

where $x(t)$ is a true solution of the equation $x' = f(t, x)$, is $O(h^3)$. [You may assume that f is C^∞ in t and x on $\mathbf{R} \times \mathbf{R^n}$ and that it is uniformly Lipschitz in x.]

Show that for $h > 0$ sufficiently small there is a unique solution x_{i+1} (for given x_i), which can be found by Picard iteration. [In practice, however, Newton's method is usually used for implicit ODE solvers.]

2. Show that the local truncation error in the linear multistep method

$$x_{i+2} - 3x_{i+1} + 2x_i = h\left[\frac{13}{12}f(t_{i+2}, x_{i+2}) - \frac{5}{3}f(t_{i+1}, x_{i+1}) - \frac{5}{12}f(t_i, x_i)\right]$$

is $O(h^3)$.

3. Show that if r is a root of multiplicity $m > 1$ of the characteristic polynomial of the linear difference equation

$$x_{i+k} + a_{k-1}x_{i+k-1} + \ldots + a_1 x_{i+1} + a_0 x_i = 0,$$

then the sequences $x_i = i^j r^i$ for $0 \leq j \leq m-1$ are solutions.

4. (2006 prelim, problem 4.) Consider the initial value ODE problem $u' = f(t, u)$, $0 \leq t \leq T$, $u(0) = u_0$, where f is C^∞ in t and u. Consider numerical methods of the form

$$u_{i+2} + a_1 u_{i+1} + a_0 u_i = hbf(t_{i+2}, u_{i+2}),$$

where u_i represents the approximate solution at $t_i = ih$, $h = T/N$.

(a) Determine the coefficients a_0, a_1, and b that give the highest order local truncation error for the method, and show what that order is.

(b) Is the resulting method *convergent* (i.e., does the approximate solution converge uniformly to the true solution on the mesh points as $h \to 0$)? Explain why or why not (i.e., either prove your answer directly or quote a theorem and show that all of the hypotheses of the theorem are satisfied).