# The scenery factor of the $[T, T^{-1}]$ transformation is not loosely Bernoulli

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**Abstract**

Kalikow proved that the $[T, T^{-1}]$ transformation is not isomorphic to a Bernoulli shift [3]. We show that the scenery factor of the $[T, T^{-1}]$ transformation is not isomorphic to a Bernoulli shift. Moreover we show that it is not Kakutani equivalent to a Bernoulli shift.

## 1 Introduction

The $[T, T^{-1}]$ transformation is a random walk on a random scenery. It is defined as follows. Let $X = \{1, -1\}^{\mathbb{Z}}$ and $Y = \{\text{red, blue}\}^{\mathbb{Z}}$. Let $\sigma$ be the left shift on $X$ ($\sigma(x)_i = x_{i+1}$) and let $T$ be the left shift on $Y$. Let $\mu'$ be (1/2,1/2) product measure on $X$ and $\mu''$ be (1/2,1/2) product measure on $Y$.

We define the transformation $[T, T^{-1}] : X \times Y \to X \times Y$ by

$$[T, T^{-1}](x, y) = \begin{cases} (\sigma(x), T(y)) & \text{if } x_0 = 1 \\ (\sigma(x), T^{-1}(y)) & \text{if } x_0 = -1. \end{cases}$$

Let $\mathcal{F}$ be the Borel $\sigma$-algebra and $\mu = \mu' \times \mu''$. Then the $[T, T^{-1}]$ transformation is the four-tuple $(X \times Y, [T, T^{-1}], \mathcal{F}, \mu)$.

The $[T, T^{-1}]$ transformation was introduced for its ergodic theoretic properties. It is easy to show that this transformation is a $K$ transformation [6]. For many years it was an open question to determine whether the $[T, T^{-1}]$ transformation is isomorphic to a Bernoulli shift. Kalikow settled the question with the following theorem [3].

**Theorem 1.** *The $[T, T^{-1}]$ transformation is not isomorphic to a Bernoulli shift. Moreover, it is not loosely Bernoulli.*

The $[T, T^{-1}]$ transformation also has probabilistic interest. Given $x$ let

$$S(i) = S_x(i) = \begin{cases} \sum_0^{i-1} x_j & \text{if } i > 0 \\ -\sum_i^{-1} x_j & \text{if } i < 0 \\ 0 & \text{if } i = 0. \end{cases}$$

Define $C(x, y)_i = y_{S(i)}$. We refer to this as the color observed at time $i$.

Probabilists have focused on two questions. The first question is of reconstruction. In this problem you are given the sequence $C(x, y)_i$, $i \geq 0$, and you are trying to reconstruct $y$. The best result for reconstruction is the following theorem by Matzinger [5].

**Theorem 2.** *There exists a function $F : X \times Y \to Y$ so that*

1. *$F(x, y) = F(x', y')$ if $C(x, y)_j = C(x', y')_j$ for all $j \geq 0$ and*

2. *there exists an even $m$ such that $F(x, y)_j = y_{j+m}$ for all $j$ or $F(x, y)_j = y_{-j+m}$ for all $j$ a.s.*

*In the course of the proof Matzinger proves the following results. There is a function $H : X \times Y \to \mathbb{Z}^{\mathbb{N}}$ and sets $D_i$ such that*

3. *for all $(x, y)$ and $i$ if $C(x', y')_j = C(x, y)_j$ for all $j \leq e^{i^4}$ then $H(x, y)_i = H(x', y')_i$,*

4. *$\lim \mu(D_i) = 1$, and*

5. *if $C(x, y)_j = C(x', y')_j$ for all $j$, there exists an even $m$ such that $y_j = y'_{j+m}$, and both $(x, y), (x', y') \in D_i$ then*

$$y_{j+S_x(H(x,y)_i)} = y'_{j+S_{x'}(H(x',y')_i)}.$$

**Note:** The last half of Theorem 2 does not appear in this form in [5]. To see how this follows we choose $D_i$ to be the set denoted by $\cap_{j \geq i} \left( E_0^j \cap E^j \right)$ in [5]. We choose $H(x, y)_i$ to be the value denoted by $t_6^i$ in [5]. Then statement 3 follows from Algorithm 7. Statement 4 follows from Lemmas 3 and 5. Statement 5 follows from Algorithms 3 and 7.

The second question of probabilistic interest is one of distinguishability. Each $y$ and $n$ determines a measure $m_{y,n}$ on $\{$red,blue$\}^{[n,\infty)}$ by

$$m_{y,n}(A) = \mu'(\{x \text{ such that } C(x, y) \in A\}).$$

Call $y$ and $y'$ **distinguishable** if $m_{y,n}$ and $m_{y',n}$ are mutually singular for all $n$. It is easy to see that if there exists an even $m$ such that $y_i = y'_{i+m}$ for all $i$ or $y_i = y'_{-i+m}$ for all $i$ then $y$ and $y'$ are not distinguishable. The following question was raised by den Hollander and Keane and independently by Benjamini and Kesten [1]. If $y$ and $y'$ are not distinguishable, does there necessarily exist an

even $m$ such that $y_i = y'_{i+m}$ for all $i$ or $y_i = y'_{-i+m}$ for all $i$? This was recently answered in the negative by Lindenstrauss [4].

In this paper we use Theorem 2 to study the ergodic theoretic properties of the $[T, T^{-1}]$ process. We call the factor that associates two points $(x, y)$ and $(x', y')$ if $C(x, y)_i = C(x', y')_i$ for all $i$ the **scenery factor**, $(X \times Y, [T, T^{-1}], \mathcal{G}, \mu)$. The main result of this paper is the following.

**Theorem 3.** *The scenery factor is not isomorphic to a Bernoulli shift. Moreover it is not loosely Bernoulli.*

Recently, Steif gave an elementary proof of a closely related theorem. He proved that the scenery factor is not a finitary factor of a Bernoulli shift [9].

## 2   Proof

The equivalence relation that associates $(x, y)$ and $(x', y')$ if

1. $C(x, y)_i = C(x', y')_i$ for all $i$ and

2. $y = T^m y'$ for some even $m$

defines a factor, $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$. The factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ is a two point extension of the scenery factor a.s. Both of these statements follow from Theorem 2.

For any partition $P$ and any $a, b \in P^{\mathbb{Z}}$ let

$$\bar{d}^P_{[0,N]}(a, b) = |\{i \; : \; i \in [0, N] \text{ and } a_i \neq b_i\}|/(N+1).$$

For any two measures $\mu$ and $\nu$ on $P^{\mathbb{Z}}$ define

$$\bar{d}^P_{[0,N]}(\mu, \nu) = \inf_m \int d_{[0,N]}(a, b) dm$$

where the infinum is taken over all joinings of $\mu$ and $\nu$. We set $P$ to be the time zero partition of $X \times Y$. A point $(x, y)$ in $X \times Y$ defines a sequence in $P^{\mathbb{Z}}$ with $i$th component $P([T, T^{-1}]^i(x, y)) = (x_i, C_i(x, y))$.

**Theorem 4.** $(X \times Y, [T, T^{-1}], \mathcal{F}, \mu)$ *is isomorphic to* $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu) \times (\Omega, \sigma, \Sigma, \nu)$*, where* $(\Omega, \sigma, \Sigma, \nu)$ *is a Bernoulli shift.*

*Proof.* An atom of $\mathcal{H}$ is given by $z, a \in \{\text{red,blue}\}^{\mathbb{Z}}$ such that there exists an $x$ so that $z_i = C(x, a)_i$ for all $i$. The atoms given by $z, a$ and $z', a'$ are equivalent if $z = z'$ and there exists an even $m$ such that $a_i = a'_{i+m}$. Given an atom $z, a$ of $\mathcal{H}$ define $\tilde{\mu}_{z,a}$ by

$$\tilde{\mu}_{z,a}(A) = \mu\{(x, y) \in A \mid C(x, y)_i = z_i \; \forall i \text{ and there exists an even } m \text{ such that } y = T^m a\}.$$

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Given $x \in X$ define $\bar{x} = \{x' \ : \ x_i = x'_i \ \forall \ i \leq 0\}$. Also define $\mu_{(x,y)}$ by

$$\mu_{(x,y)}(A) = \mu\{(x',y') \in A \mid x' \in \bar{x} \text{ and } C(x,y)_i = C(x',y')_i \ \forall i\}.$$

By Thouvenot's relative isomorphism theorem, the theorem is equivalent to checking that the $[T,T^{-1}]$ transformation is **relatively very weak Bernoulli** with respect to $(X \times Y, [T,T^{-1}], \mathcal{H}, \mu)$ [10] [7]. This means that given almost every atom $z,a$ of $\mathcal{H}$ and any $\epsilon > 0$ there exists an $N$ and a set $G$ such that

1. $\tilde{\mu}_{z,a}(G) > 1 - \epsilon$ and

2. for any $(x,y),(x',y') \in G$
$$\bar{d}^P_{[0,N]}(\mu_{(x,y)}, \mu_{(x',y')}) < \epsilon.$$

Fix $z$ and $a$. Let

$$S_M = \left\{ (x,y) : \ \mu_{(x,y)}\left\{(\tilde{x},\tilde{y}) : \ (\tilde{x},\tilde{y}) \in D_M\right\} > 1 - \epsilon \right\}.$$

Let $M$ be such that $\tilde{\mu}_{z,a}(S_M) > 1 - \epsilon$. This exists for almost every atom by Theorem 2. Now let $G$ be $S_M$ restricted to the atom defined by $z,a$.

Let $(c,d),(e,f) \in D_M$ both be points in the atom determined by $z$ and $a$. By Line 5 of Theorem 2 we have that

$$d_{j+S(H(c,d)_M)} = f_{j+S(H(e,f)_M)}$$

Thus for any $(x,y),(x',y') \in G$ and any joining $\gamma$ of $\mu_{(x,y)}$ and $\mu_{(x',y')}$ we have

$$\gamma\{(c,d),(e,f) : \ d_{j+S(H(c,d)_M)} = f_{j+S(H(e,f)_M)} \text{ for all } j\} > 1 - 2\epsilon.$$

Consider a joining $\Gamma$ of $\mu_{(x,y)}$ and $\mu_{(x',y')}$ such that $c_i = e_i$ for all $i > H(c,d)_M = H(e,f)_M$. We get that

$$\Gamma\left\{(c,d),(e,f) : C(c,d)_j = C(e,f)_j \text{ for all } j > H(c,d)_M\right\} > 1 - 2\epsilon.$$

Let $N > e^{M^4}/\epsilon > H(c,d)_M/\epsilon$. Thus the joining $\Gamma$ shows that

$$\bar{d}^P_{[0,N]}(\mu_{(x,y)}, \mu_{(x',y')}) < 3\epsilon.$$

$\square$

**Proof of Theorem 3:** By Theorem 1 the $[T,T^{-1}]$ transformation is not isomorphic to a Bernoulli shift [3]. By Theorem 4 the $[T,T^{-1}]$ transformation is the direct product of the factor $(X \times Y, [T,T^{-1}], \mathcal{H}, \mu)$ with a Bernoulli shift. Thus the factor $(X \times Y, [T,T^{-1}], \mathcal{H}, \mu)$ is not isomorphic to a Bernoulli shift. By [3] the $[T,T^{-1}]$ transformation is not loosely Bernoulli. As the direct

product of a loosely Bernoulli transformation and a Bernoulli shift is loosely Bernoulli, the factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ is not loosely Bernoulli either.

The factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ is a two point extension of the scenery factor. It is weak mixing since it is the factor of the $[T, T^{-1}]$ transformation which is K (and thus weak mixing). The two point extension of a Bernoulli shift which is weak mixing is isomorphic to a Bernoulli shift [8]. Thus scenery factor is not isomorphic to a Bernoulli shift.

Similarly we can show that the scenery factor is not loosely Bernoulli. The factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ is not loosely Bernoulli. The two point extension of a loosely Bernoulli transformation is loosely Bernoulli [8]. Thus if the scenery factor were loosely Bernoulli then the factor $(X \times Y, [T, T^{-1}], \mathcal{H}, \mu)$ would be as well. This can not be, so the scenery factor is not loosely Bernoulli and is not Kakutani equivalent to a Bernoulli shift [2]. □

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