Axioms of Incidence Geometry

Incidence Axiom 1. *There exist at least three distinct noncollinear points.*

Incidence Axiom 2. *Given any two distinct points, there is at least one line that contains both of them.*

Incidence Axiom 3. *Given any two distinct points, there is at most one line that contains both of them.*

Incidence Axiom 4. *Given any line, there are at least two distinct points that lie on it.*

Theorems of Incidence Geometry

Theorem 2.25. *Given any point A, there exists another point that is distinct from A.*

Theorem 2.26. *Given any point, there exists a line that contains it.*

Corollary 2.27. *If A and B are points (not necessarily distinct), there is a line that contains both of them.*

Theorem 2.28. *If ℓ is a line and A and B are two distinct points on ℓ, then $\overleftrightarrow{AB} = \ell$.*

Theorem 2.29. *If A and B are distinct points, and C is any point that does not lie on $\overleftrightarrow{AB}$, then A, B, and C are noncollinear.*

Theorem 2.30. *If A, B, and C are noncollinear points, then A and B are distinct, and C does not lie on $\overleftrightarrow{AB}$.*

Corollary 2.31. *If A, B, and C are noncollinear points, then A, B, and C are all distinct. Moreover, A does not lie on $\overleftrightarrow{BC}$, B does not lie on $\overleftrightarrow{AC}$, and C does not lie on $\overleftrightarrow{AB}$.*

Theorem 2.32. *Given a line ℓ and a point A that lies on ℓ, there exists a point B that lies on ℓ and is distinct from A.*

Theorem 2.33. *Given any line, there exists a point that does not lie on it.*

Theorem 2.34. *Given two distinct points A and B, there exists a point C such that A, B, and C are noncollinear.*

Theorem 2.35. *Given any point A, there exist points B and C such that A, B, and C are noncollinear.*

Theorem 2.36. *Given two distinct points A and B, there exists a line that contains A but not B.*

Theorem 2.37. *Given any point, there exists a line that does not contain it.*

Theorem 2.38. *If A, B, and C are noncollinear points, then $\overleftrightarrow{AB} \neq \overleftrightarrow{AC}$.*

Corollary 2.39. *If A, B, and C are noncollinear points, then $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$ are all distinct.*

Theorem 2.40. *If A, B, and C are collinear points, and neither B nor C is equal to A, then $\overleftrightarrow{AB} = \overleftrightarrow{AC}$.*

Theorem 2.41. *Given two distinct, nonparallel lines, there exists a unique point that lies on both of them.*

Theorem 2.42. *Given any point, there are at least two distinct lines that contain it.*

Postulates of Neutral Geometry

Postulate 1 (The Set Postulate). *Every line is a set of points, and there is a set of all points called* ***the plane****.*

Postulate 2 (The Existence Postulate). *There exist at least three distinct noncollinear points.*

Postulate 3 (The Unique Line Postulate). *Given any two distinct points, there is a unique line that contains both of them.*

Postulate 4 (The Distance Postulate). *For every pair of points A and B, the distance from A to B is a nonnegative real number determined by A and B.*

Postulate 5 (The Ruler Postulate). *For every line ℓ, there is a bijective function $f : \ell \to \mathbb{R}$ with the property that for any two points $A, B \in \ell$, we have $AB = |f(B) - f(A)|$.*

Postulate 6 (The Plane Separation Postulate). *For any line ℓ, the set of all points not on ℓ is the union of two disjoint subsets called the* ***sides of ℓ****. If A and B are distinct points not on ℓ, then A and B are on the same side of ℓ if and only if $\overline{AB} \cap \ell = \varnothing$.*

Postulate 7 (The Angle Measure Postulate). *For every angle $\angle ab$, the measure of $\angle ab$ is a real number in the closed interval $[0, 180]$ determined by $\angle ab$.*

Postulate 8 (The Protractor Postulate). *For every ray $\overrightarrow{r}$ and every point P not on $\overleftrightarrow{r}$, there is a bijective function $g\colon \operatorname{HR}(\overrightarrow{r}, P) \to [0, 180]$ that assigns the number 0 to $\overrightarrow{r}$ and the number 180 to the ray opposite $\overrightarrow{r}$, and such that if $\overrightarrow{a}$ and $\overrightarrow{b}$ are any two rays in $\operatorname{HR}(\overrightarrow{r}, P)$, then $m\angle ab = \left| g\left(\overrightarrow{b}\right) - g\left(\overrightarrow{a}\right)\right|$.*

Postulate 9 (The SAS Postulate). *If there is a correspondence between the vertices of two triangles such that two sides and the included angle of one triangle are congruent to the corresponding sides and angle of the other triangle, then the triangles are congruent under that correspondence.*

Theorems of Neutral Geometry

All of the theorems of incidence geometry.

Theorem 3.1. *Every line contains infinitely many distinct points.*

Corollary 3.2 (Incidence Axiom 4). *Given any line, there are at least two distinct points that lie on it.*

Lemma 3.3 (Ruler Sliding Lemma). *Suppose ℓ is a line and $f : \ell \to \mathbb{R}$ is a coordinate function for ℓ. Given a real number c, define a new function $f_1 : \ell \to \mathbb{R}$ by $f_1(X) = f(X) + c$ for all $X \in \ell$. Then f_1 is also a coordinate function for ℓ.*

Lemma 3.4 (Ruler Flipping Lemma). *Suppose ℓ is a line and $f : \ell \to \mathbb{R}$ is a coordinate function for ℓ. If we define $f_2 : \ell \to \mathbb{R}$ by $f_2(X) = -f(X)$ for all $X \in \ell$, then f_2 is also a coordinate function for ℓ.*

Theorem 3.5 (Ruler Placement Theorem). *Suppose ℓ is a line and A, B are two distinct points on ℓ. Then there exists a coordinate function $f : \ell \to \mathbb{R}$ such that $f(A) = 0$ and $f(B) > 0$.*

Theorem 3.6 (Properties of Distances). *If A and B are any two points, their distance has the following properties:*

(*a*) *$AB = BA$.*

(*b*) *$AB = 0$ if and only if $A = B$.*

(*c*) *$AB > 0$ if and only if $A \neq B$.*

Theorem 3.7 (Symmetry of Betweenness of Points). *If A, B, C are any three points, then $A * B * C$ if and only if $C * B * A$.*

Theorem 3.8 (Betweenness Theorem for Points). *Suppose A, B, and C are points. If $A * B * C$, then $AB + BC = AC$.*

Theorem 3.9 (Hilbert's Betweenness Axiom). *Given three distinct collinear points, exactly one of them lies between the other two.*

Corollary 3.10 (Consistency of Betweenness of Points). *Suppose A, B, C are three points on a line ℓ. Then $A * B * C$ if and only if $f(A) * f(B) * f(C)$ for every coordinate function $f : \ell \to \mathbb{R}$.*

Theorem 3.11 (Partial Converse to the Betweenness Theorem for Points). *If A, B, and C are three distinct collinear points such that $AB + BC = AC$, then $A * B * C$.*

Theorem 3.12. *Suppose A and B are distinct points. Then $\overleftrightarrow{AB} = \{P : P * A * B$ or $P = A$ or $A * P * B$ or $P = B$ or $A * B * P\}$.*

Lemma 3.13. *If $A_1, A_2, \dots, A_k$ are distinct collinear points, then $A_1 * A_2 * \cdots * A_k$ if and only if $A_k * \dots A_2 * A_1$.*

Theorem 3.14. *Given any k distinct collinear points, they can be labeled $A_1, \dots, A_k$ in some order such that $A_1 * A_2 * \cdots * A_k$.*

Theorem 3.15. *Suppose A, B, C are points such that $A*B*C$. If P is any point on $\overleftrightarrow{AB}$, then one and only one of the following relations holds: $P=A$, $P=B$, $P=C$, $P*A*B*C$, $A*P*B*C$, $A*B*P*C$, $A*B*C*P$.*

Theorem 3.16. *Suppose A, B, C, D are four distinct points. If any of the following pairs of conditions holds, then $A*B*C*D$:*

$$\begin{aligned} &A*B*C \text{ and } B*C*D; \qquad \text{or}\\ &A*B*C \text{ and } A*C*D; \qquad \text{or}\\ &A*B*D \text{ and } B*C*D. \end{aligned}$$

On the other hand, if $A*B*C*D$, then all of the following conditions are true:

$$A*B*C, \qquad A*B*D, \qquad A*C*D, \qquad \text{and} \qquad B*C*D.$$

Lemma 3.17. *If A and B are two distinct points, then $\overline{AB} \subseteq \overleftrightarrow{AB}$.*

Theorem 3.18 (Segment Extension Theorem). *If $\overline{AB}$ is any segment, there exist points $C, D \in \overleftrightarrow{AB}$ such that $C*A*B$ and $A*B*D$.*

Lemma 3.19. *Suppose S and T are sets of points in the plane with $S \subseteq T$. Every passing point of S is also a passing point of T.*

Theorem 3.20. *Suppose A and B are distinct points. Then A and B are extreme points of $\overline{AB}$, and every other point of $\overline{AB}$ is a passing point.*

Corollary 3.21 (Consistency of Endpoints of Segments). *Suppose A and B are distinct points, and C and D are distinct points, such that $\overline{AB} = \overline{CD}$. Then either $A=C$ and $B=D$, or $A=D$ and $B=C$.*

Theorem 3.22 (Euclid's Common Notions for Segments).

- (a) **Transitive Property of Congruence:** *Two segments that are both congruent to a third segment are congruent to each other.*
- (b) **Segment Addition Theorem:** *Suppose A, B, C, A', B', C' are points such that $A*B*C$ and $A'*B'*C'$. If $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.*
- (c) **Segment Subtraction Theorem:** *Suppose A, B, C, A', B', C' are points such that $A*B*C$ and $A'*B'*C'$. If $\overline{AC} \cong \overline{A'C'}$ and $\overline{AB} \cong \overline{A'B'}$, then $\overline{BC} \cong \overline{B'C'}$.*
- (d) **Reflexive Property of Congruence:** *Every segment is congruent to itself.*
- (e) **The Whole Segment is Greater Than the Part:** *If A, B, and C are points such that $A*B*C$, then $AC > AB$.*

Lemma 3.23 (Coordinate Representation of a Segment). *Suppose A and B are distinct points, and $f: \overleftrightarrow{AB} \to \mathbb{R}$ is a coordinate function for $\overleftrightarrow{AB}$. Then*

$$\begin{aligned} \overline{AB} &= \{P \in \overleftrightarrow{AB} : f(A) \le f(P) \le f(B)\} \qquad \text{if } f(A) < f(B);\\ \overline{AB} &= \{P \in \overleftrightarrow{AB} : f(A) \ge f(P) \ge f(B)\} \qquad \text{if } f(A) > f(B). \end{aligned}$$

Theorem 3.24. *If A, B, C are points such that $A*B*C$, then the following set equalities hold:*

- (a) $\overline{AB} \cup \overline{BC} = \overline{AC}$.
- (b) $\overline{AB} \cap \overline{BC} = \{B\}$.

Corollary 3.25. *If $A*B*C$, then $\overline{AB} \subseteq \overline{AC}$ and $\overline{BC} \subseteq \overline{AC}$.*

Lemma 3.26. *Let $\overline{AB}$ be a segment, and let M be a point. The following statements are all equivalent to each other:*

- (a) *M is a midpoint of $\overline{AB}$ (i.e., $M \in \operatorname{Int} \overline{AB}$ and $MA = MB$).*
- (b) *$M \in \overleftrightarrow{AB}$ and $MA = MB$.*
- (c) *$M \in \overline{AB}$ and $AM = \frac{1}{2}AB$.*

Theorem 3.27 (Existence and Uniqueness of Midpoints). *Every segment has a unique midpoint.*

Theorem 3.28. *Every segment contains infinitely many distinct points.*

Theorem 3.29 (Euclid's Postulate 3). *Given two distinct points O and A, there exists a circle whose center is O and whose radius is OA.*

Lemma 3.30. *Suppose A and B are distinct points, and P is a point on the line $\overleftrightarrow{AB}$. Then $P \notin \overrightarrow{AB}$ if and only if $P*A*B$.*

Lemma 3.31. *Suppose A and B are distinct points. Then $\overline{AB} \subseteq \overrightarrow{AB} \subseteq \overleftrightarrow{AB}$.*

Lemma 3.32 (Coordinate Representation of a Ray). *Suppose A and B are distinct points, and $f: \overleftrightarrow{AB} \to \mathbb{R}$ is a coordinate function for $\overleftrightarrow{AB}$. Then*

$$\begin{aligned} \overrightarrow{AB} &= \{P \in \overleftrightarrow{AB} : f(P) \ge f(A)\} \qquad \text{if } f(A) < f(B);\\ \overrightarrow{AB} &= \{P \in \overleftrightarrow{AB} : f(P) \le f(A)\} \qquad \text{if } f(A) > f(B). \end{aligned}$$

Lemma 3.33 (Representation of a Ray in Adapted Coordinates). *Suppose A and B are distinct points, and $f : \overleftrightarrow{AB} \to \mathbb{R}$ is a coordinate function adapted to $\overrightarrow{AB}$. If P is any point on $\overleftrightarrow{AB}$, then $P \in \overrightarrow{AB}$ if and only if $f(P) \geq 0$, and $P \in \operatorname{Int} \overrightarrow{AB}$ if and only if $f(P) > 0$.*

Lemma 3.34 (Ordering Lemma for Points). *Suppose $\overrightarrow{a}$ is a ray starting at a point A, and B and C are interior points of $\overrightarrow{a}$ such that $AC > AB$. Then $A * B * C$.*

Theorem 3.35 (Segment Construction Theorem). *Suppose $\overrightarrow{a}$ is a ray starting at a point A, and r is a positive real number. Then there exists a unique point C in the interior of $\overrightarrow{a}$ such that $AC = r$.*

Corollary 3.36 (Unique Point Theorem). *Suppose $\overrightarrow{a}$ is a ray starting at a point A, and C and C' are points in $\operatorname{Int} \overrightarrow{a}$ such that $AC = AC'$. Then $C = C'$.*

Corollary 3.37 (Euclid's Segment Cutoff Theorem). *If $\overline{AB}$ and $\overline{CD}$ are segments with $CD > AB$, there is a unique point E in the interior of $\overline{CD}$ such that $\overline{CE} \cong \overline{AB}$.*

Theorem 3.38 (Rays with the Same Endpoint). *Suppose $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are rays with the same endpoint.*

(*a*) *If A, B, and C are collinear, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear.*

(*b*) *If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear, then they are either equal or opposite, but not both.*

(*c*) *If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, then $\overrightarrow{AB} \cap \overrightarrow{AC} = \{A\}$ and $\overrightarrow{AB} \cup \overrightarrow{AC} = \overleftrightarrow{AC}$.*

(*d*) *$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are equal if and only if they have an interior point in common.*

(*e*) *$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays if and only if $C * A * B$.*

Theorem 3.39 (Opposite Ray Theorem). *Every ray has a unique opposite ray.*

Theorem 3.40. *Let $\overrightarrow{AB}$ be the ray from A through B. Then A is the only extreme point of $\overrightarrow{AB}$.*

Corollary 3.41 (Consistency of Endpoints of Rays). *If A, B are distinct points and C, D are distinct points such that $\overrightarrow{AB} = \overrightarrow{CD}$, then $A = C$.*

Theorem 3.42. *Suppose A and B are two distinct points. Then the following set equalities hold:*

(*a*) $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$.

(*b*) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

Theorem 3.43 (Properties of Sides of Lines). *Suppose ℓ is a line.*

(*a*) *Both sides of ℓ are nonempty sets.*

(*b*) *If A and B are distinct points not on ℓ, then A and B are on opposite sides of ℓ if and only if $\overline{AB} \cap \ell \neq \varnothing$.*

Lemma 3.44 (The Y-Lemma). *Suppose ℓ is a line, A is a point on ℓ, and B is a point not on ℓ. Then every interior point of $\overrightarrow{AB}$ is on the same side of ℓ as B, and $\overrightarrow{AB} \subseteq \operatorname{CHP}(\ell, B)$.*

Lemma 3.45 (The X-Lemma). *Suppose $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are opposite rays, and ℓ is a line that intersects $\overleftrightarrow{AB}$ only at O. Then $\overrightarrow{OA}$ and $\overrightarrow{OB}$ lie on opposite sides of ℓ.*

Theorem 3.46. *Suppose ℓ is a line, A is a point on ℓ, and B is a point not on ℓ. Then $\overrightarrow{AB} = \overleftrightarrow{AB} \cap \operatorname{CHP}(\ell, B)$.*

Theorem 3.47. *If $S_1, \dots, S_k$ are convex subsets of the plane, then $S_1 \cap \dots \cap S_k$ is convex.*

Theorem 3.48. *Every line is a convex set.*

Theorem 3.49. *Every segment is a convex set.*

Theorem 3.50. *Every open or closed half-plane is a convex set.*

Theorem 3.51. *Every ray is a convex set.*

Theorem 4.1. *If $\angle ab$ is a proper angle, then the common endpoint of $\overrightarrow{a}$ and $\overrightarrow{b}$ is the only extreme point of $\angle ab$.*

Corollary 4.2 (Consistency of Vertices of Proper Angles). *If $\angle AOB$ and $\angle A'O'B'$ are equal proper angles, then $O = O'$.*

Theorem 4.3 (Properties of Angle Measures). *Suppose $\angle ab$ is any angle.*

(*a*) $m\angle ab = m\angle ba$.
(*b*) $m\angle ab = 0°$ *if and only if $\angle ab$ is a zero angle.*
(*c*) $m\angle ab = 180°$ *if and only if $\angle ab$ is a straight angle.*
(*d*) $0° < m\angle ab < 180°$ *if and only if $\angle ab$ is a proper angle.*

Theorem 4.4 (Euclid's Postulate 4). *All right angles are congruent.*

Theorem 4.5 (Angle Construction Theorem). *Let O be a point, let $\overrightarrow{a}$ be a ray starting at O, and let x be a real number such that $0 < x < 180$. On each side of $\overleftrightarrow{a}$, there is a unique ray $\overrightarrow{b}$ starting at O such that $m\angle ab = x$.*

Corollary 4.6 (Unique Ray Theorem). *Let $\overrightarrow{a}$ be a ray starting at a point O. If $\overrightarrow{b}$ and $\overrightarrow{b}'$ are rays starting at O and lying on the same side of $\overleftrightarrow{a}$ such that $m\angle ab = m\angle ab'$, then $\overrightarrow{b}$ and $\overrightarrow{b}'$ are the same ray.*

Theorem 4.7 (Symmetry of Betweenness of Rays). *If $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are rays with a common endpoint, then $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$ if and only if $\overrightarrow{c} * \overrightarrow{b} * \overrightarrow{a}$.*

Theorem 4.8 (Betweenness Theorem for Rays). *If $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are rays such that $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$, then $m\angle ab + m\angle bc = m\angle ac$.*

Theorem 4.9. *If $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are rays with a common endpoint, no two of which are collinear, and all lying in a single half-rotation, then exactly one of them lies between the other two.*

Corollary 4.10 (Consistency of Betweenness of Rays). *Let $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ be distinct rays with a common endpoint, no two of which are collinear, and all lying in a single half-rotation. If $\mathrm{HR}(\overrightarrow{r}, P)$ is any half-rotation containing all three rays and g is a corresponding coordinate function, then $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$ if and only if $g(\overrightarrow{a}) * g\left(\overrightarrow{b}\right) * g(\overrightarrow{c})$.*

Theorem 4.11 (Euclid's Common Notions for Angles).

(*a*) **Transitive Property of Congruence:** *Two angles that are both congruent to a third angle are congruent to each other.*

(*b*) **Angle Addition Theorem:** *Suppose $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{a}', \overrightarrow{b}', \overrightarrow{c}'$ are rays such that $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$ and $\overrightarrow{a}' * \overrightarrow{b}' * \overrightarrow{c}'$. If $\angle ab \cong \angle a'b'$ and $\angle bc \cong \angle b'c'$, then $\angle ac \cong \angle a'c'$.*

(*c*) **Angle Subtraction Theorem:** *Suppose $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{a}', \overrightarrow{b}', \overrightarrow{c}'$ are rays such that $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$ and $\overrightarrow{a}' * \overrightarrow{b}' * \overrightarrow{c}'$. If $\angle ac \cong \angle a'c'$ and $\angle ab \cong \angle a'b'$, then $\angle bc \cong \angle b'c'$.*

(*d*) **Reflexive Property of Congruence:** *Every angle is congruent to itself.*

(*e*) **The Whole Angle is Greater Than the Part:** *If $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are rays such that $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$, then $m\angle ac > m\angle ab$.*

Theorem 4.12. *If $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are rays such that $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$, then $\angle ab$ and $\angle bc$ are adjacent angles.*

Theorem 4.13. *Supplements of congruent angles are congruent, and complements of congruent angles are congruent.*

Theorem 4.14 (Linear Pair Theorem). *If two angles form a linear pair, then they are supplementary.*

Corollary 4.15. *If two angles in a linear pair are congruent, then they are both right angles.*

Theorem 4.16 (Partial Converse to the Linear Pair Theorem). *If two adjacent angles are supplementary, then they form a linear pair.*

Theorem 4.17 (Vertical Angles Theorem). *Vertical angles are congruent.*

Theorem 4.18 (Partial Converse to the Vertical Angles Theorem). *Suppose $\overrightarrow{a}$ and $\overrightarrow{c}$ are opposite rays starting at a point O, and $\overrightarrow{b}$ and $\overrightarrow{d}$ are rays starting at O and lying on opposite sides of $\overleftrightarrow{a}$. If $\angle ab \cong \angle cd$, then $\overrightarrow{b}$ and $\overrightarrow{d}$ are opposite rays.*

Theorem 4.19 (Linear Triple Theorem). *If $\angle ab$, $\angle bc$, and $\angle cd$ form a linear triple, then their measures add up to $180°$.*

Theorem 4.20. *The interior of a proper angle is a convex set.*

Lemma 4.21. *Suppose $\angle AOC$ is a proper angle, and $\overrightarrow{OB}$ is a ray that lies in the interior of $\angle AOC$. Then $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$.*

Theorem 4.22 (The 360 Theorem). *Suppose $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are three distinct rays with the same endpoint, such that no two of the rays are collinear and none of the rays lies in the interior of the angle formed by the other two. Then $m\angle ab + m\angle bc + m\angle ac = 360°$.*

Lemma 4.23 (Interior Lemma). *Suppose $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are three rays with the same endpoint, no two of which are collinear. Then $\overrightarrow{b}$ lies in the interior of $\angle ac$ if and only if $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$.*

Corollary 4.24 (Restatement of the 360 Theorem in Terms of Betweenness). *Suppose $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are three distinct rays with the same endpoint, such that no two of the rays are collinear and none of the rays lies between the other two. Then $m\angle ab + m\angle bc + m\angle ac = 360°$.*

Lemma 4.25 (Ordering Lemma for Rays). *Suppose $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are rays with the same endpoint, such that $\overrightarrow{b}$ and $\overrightarrow{c}$ are on the same side of $\overleftrightarrow{a}$ and $m\angle ab < m\angle ac$. Then $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$.*

Lemma 4.26 (Adjacency Lemma). *Suppose $\angle ab$ and $\angle bc$ are adjacent angles sharing the common side $\overrightarrow{b}$. If either of the following conditions holds, then $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$.*

(*a*) *$\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ all lie in a single half-rotation.*

(*b*) *$m\angle ab + m\angle bc < 180°$.*

Theorem 4.27 (Betweenness vs. Betweenness). *Suppose ℓ is a line, O is a point not on ℓ, and A, B, C are three distinct points on ℓ. Then $A * B * C$ if and only if $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$.*

Theorem 4.28 (Existence and Uniqueness of Angle Bisectors). *Every proper angle has a unique angle bisector.*

Theorem 4.29 (Four Right Angles Theorem). *If ℓ and m are perpendicular lines, then ℓ and m form four right angles.*

Theorem 4.30 (Constructing a Perpendicular). *Let ℓ be a line and let P be a point on ℓ. Then there exists a unique line m that is perpendicular to ℓ at P.*

Theorem 5.1 (Consistency of Triangle Vertices). *If $\triangle ABC$ is a triangle, the only extreme points of $\triangle ABC$ are A, B, and C. Thus if $\triangle ABC = \triangle A'B'C'$, then the sets $\{A, B, C\}$ and $\{A', B', C'\}$ are equal.*

Theorem 5.2 (Pasch's Theorem). *Suppose $\triangle ABC$ is a triangle and ℓ is a line that does not contain any of the points A, B, or C. If ℓ intersects one of the sides of $\triangle ABC$, then it also intersects another side.*

Corollary 5.3. *If $\triangle ABC$ is a triangle and ℓ is a line that does not contain any of the points A, B, or C, then either ℓ intersects exactly two sides of $\triangle ABC$ or it intersects none of them.*

Theorem 5.4 (The Crossbar Theorem). *Suppose $\triangle ABC$ is a triangle and $\overrightarrow{AD}$ is a ray between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then the interior of $\overrightarrow{AD}$ intersects the interior of $\overline{BC}$.*

Theorem 5.5 (Transitive Property of Congruence of Triangles). *Two triangles that are both congruent to a third triangle are congruent to each other.*

Theorem 5.6 (ASA Congruence). *If there is a correspondence between the vertices of two triangles such that two angles and the included side of one triangle are congruent to the corresponding angles and side of the other triangle, then the triangles are congruent under that correspondence.*

Theorem 5.7 (Isosceles Triangle Theorem). *If two sides of a triangle are congruent to each other, then the angles opposite those sides are congruent.*

Theorem 5.8 (Converse to the Isosceles Triangle Theorem). *If two angles of a triangle are congruent to each other, then the sides opposite those angles are congruent.*

Corollary 5.9. *A triangle is equilateral if and only if it is equiangular.*

Theorem 5.10 (Triangle Copying Theorem). *Suppose $\triangle ABC$ is a triangle, and $\overline{DE}$ is a segment congruent to $\overline{AB}$. On each side of $\overleftrightarrow{DE}$, there is a point F such that $\triangle DEF \cong \triangle ABC$.*

Theorem 5.11 (Unique Triangle Theorem). *Suppose $\overline{DE}$ is a segment, and F and F' are points on the same side of $\overleftrightarrow{DE}$ such that $\triangle DEF \cong \triangle DEF'$. Then $F = F'$.*

Theorem 5.12 (SSS Congruence). *If there is a correspondence between the vertices of two triangles such that all three sides of one triangle are congruent to the corresponding sides of the other triangle, then the triangles are congruent under that correspondence.*

Theorem 5.13 (Exterior Angle Inequality). *The measure of an exterior angle of a triangle is strictly greater than the measure of either remote interior angle.*

Corollary 5.14. *The sum of the measures of any two angles of a triangle is less than $180°$.*

Corollary 5.15. *Every triangle has at least two acute angles.*

Theorem 5.16 (Scalene Inequality). *Let $\triangle ABC$ be a triangle. Then $AC > BC$ if and only if $m\angle B > m\angle A$.*

Corollary 5.17. *In any right triangle, the hypotenuse is strictly longer than either leg.*

Theorem 5.18 (Triangle Inequality). *If A, B, and C are noncollinear points, then $AC < AB + BC$.*

Theorem 5.19 (General Triangle Inequality).

(*a*) *If A, B, C are any three points (not necessarily distinct), then $AC \le AB + BC$, and equality holds if and only if $A = B$, $B = C$, or $A * B * C$.*

(*b*) *If $n \ge 3$ and $A_1, \dots, A_n$ are any n points (not necessarily distinct), then $A_1A_n \le A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n$.*

Corollary 5.20 (Converse to the Betweenness Theorem for Points). *If A, B, and C are three distinct points and $AB + BC = AC$, then $A * B * C$.*

Theorem 5.21 (Hinge Theorem). *Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$. Then $m\angle A > m\angle D$ if and only if $BC > EF$.*

Theorem 5.22 (AAS Congruence). *If there is a correspondence between the vertices of two triangles such that two angles and a nonincluded side of one triangle are congruent to the corresponding angles and side of the other triangle, then the triangles are congruent under that correspondence.*

Theorem 5.24 (ASS Alternative Theorem). *Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ (the hypotheses of ASS). Then $\angle C$ and $\angle F$ are either congruent or supplementary.*

Theorem 5.25 (Angle-Side-Longer-Side Congruence). *Suppose $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$ (the hypotheses of ASS), and assume in addition that $BC \geq AB$. Then $\triangle ABC \cong \triangle DEF$.*

Theorem 5.26 (HL Congruence). *If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and one leg of another, then the triangles are congruent under that correspondence.*

Theorem 7.1 (Dropping a Perpendicular). *Suppose ℓ is a line and A is a point not on ℓ. Then there exists a unique line that contains A and is perpendicular to ℓ.*

Theorem 7.2. *Suppose $\overrightarrow{AB}$ is a ray, P is a point not on $\overleftrightarrow{AB}$, and F is the foot of the perpendicular from P to $\overleftrightarrow{AB}$.*

(a) *$F = A$ if and only if $\angle PAB$ is a right angle.*

(b) *$F \in \operatorname{Int}\overrightarrow{AB}$ if and only if $\angle PAB$ is acute.*

(c) *F lies in the interior of the ray opposite $\overrightarrow{AB}$ if and only if $\angle PAB$ is obtuse.*

Theorem 7.3. *Let $\triangle ABC$ be a triangle, and let F be the foot of the altitude from A to $\overline{BC}$.*

(a) *$F = B$ if and only if $\angle B$ is right, and $F = C$ if and only if $\angle C$ is right.*

(b) *$B * F * C$ if and only if $\angle B$ and $\angle C$ are both acute.*

(c) *$F * B * C$ if and only if $\angle B$ is obtuse, and $B * C * F$ if and only if $\angle C$ is obtuse.*

Corollary 7.4. *In any triangle, the altitude to the longest side always intersects the interior of that side.*

Corollary 7.5. *In a right triangle, the altitude to the hypotenuse always intersects the interior of the hypotenuse.*

Theorem 7.6 (Isosceles Triangle Altitude Theorem). *The altitude to the base of an isosceles triangle is also the median to the base, and is contained in the bisector of the angle opposite the base.*

Theorem 7.7 (Existence and Uniqueness of a Perpendicular Bisector). *Every segment has a unique perpendicular bisector.*

Theorem 7.8 (Perpendicular Bisector Theorem). *If $\overline{AB}$ is any segment, every point on the perpendicular bisector of $\overline{AB}$ is equidistant from A and B.*

Theorem 7.9 (Converse to the Perpendicular Bisector Theorem). *Suppose $\overline{AB}$ is a segment and P is a point that is equidistant from A and B. Then P lies on the perpendicular bisector of $\overline{AB}$.*

Theorem 7.10 (Reflection Across a Line). *Let ℓ be a line and let A be a point not on ℓ. Then there is a unique point A', called the* ***reflection of A across ℓ****, such that ℓ is the perpendicular bisector of $\overline{AA'}$.*

Lemma 7.12 (Properties of Closest Points). *Let P be a point and S be any set of points.*

(a) *If C is a closest point to P in S, then another point $C' \in S$ is also a closest point to P if and only if $PC' = PC$.*

(b) *If C is a point in S such that $PX > PC$ for every point $X \in S$ other than C, then C is the unique closest point to P in S.*

Theorem 7.13 (Closest Point on a Line). *Suppose ℓ is a line, P is a point not on ℓ, and F is the foot of the perpendicular from P to ℓ.*

(a) *F is the unique closest point to P on ℓ.*

(b) *If A and B are points on ℓ such that $F * A * B$, then $PB > PA$.*

Theorem 7.14 (Closest Point on a Segment). *Suppose $\overline{AB}$ is a segment and P is any point. Then there is a unique closest point to P in $\overline{AB}$.*

Theorem 7.15 (Angle Bisector Theorem). *Suppose $\angle AOB$ is a proper angle and P is a point on the bisector of $\angle AOB$. Then P is equidistant from $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$.*

Lemma 7.16. *Suppose $\angle AOB$ is a proper angle, and P is a point in $\operatorname{Int}\angle AOB$ that is equidistant from $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$. Then the feet of the perpendiculars from P to $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ lie in the interiors of the rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively.*

Theorem 7.17 (Partial Converse to the Angle Bisector Theorem). *Suppose $\angle AOB$ is a proper angle. If P is a point in the interior of $\angle AOB$ that is equidistant from $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$, then P lies on the angle bisector of $\angle AOB$.*

Lemma 7.18. *Suppose ℓ is a line, and S is a segment, ray, or line that is parallel to ℓ. Then all points of S lie on the same side of ℓ.*

Theorem 7.19 (Alternate Interior Angles Theorem). *If two lines are cut by a transversal making a pair of congruent alternate interior angles, then they are parallel.*

Corollary 7.20 (Corresponding Angles Theorem). *If two lines are cut by a transversal making a pair of congruent corresponding angles, then they are parallel.*

Corollary 7.21 (Consecutive Interior Angles Theorem). *If two lines are cut by a transversal making a pair of supplementary consecutive interior angles, then they are parallel.*

Corollary 7.22 (Common Perpendicular Theorem). *If two distinct lines have a common perpendicular* (*i.e., a line that is perpendicular to both*)*, then they are parallel.*

Theorem 7.23 (Two-Point Equidistance Theorem). *Suppose ℓ and m are two distinct lines, and there exist two distinct points on ℓ that are on the same side of m and equidistant from m. Then $\ell \parallel m$.*

Corollary 7.24 (Equidistance Theorem). *If one of two distinct lines is equidistant from the other, then they are parallel.*

Theorem 7.25 (Existence of Parallels). *For every line ℓ and every point A that does not lie on ℓ, there exists a line m such that A lies on m and $m \parallel \ell$. It can be chosen so that ℓ and m have a common perpendicular that contains A.*

Theorem 8.1 (Consistency of Polygon Vertices). *Suppose $\mathcal{P} = A_1 \dots A_n$ and $\mathcal{Q} = B_1 \dots B_m$ are polygons such that $\mathcal{P} = \mathcal{Q}$. Then the two sets of vertices $\{A_1, \dots, A_n\}$ and $\{B_1, \dots, B_m\}$ are equal.*

Lemma 8.2 (Edge-Line Lemma). *If $\mathcal{P}$ is a convex polygon and $\overline{AB}$ is an edge of $\mathcal{P}$, then $\mathcal{P} \cap \overleftrightarrow{AB} = \overline{AB}$.*

Theorem 8.3. *If $\mathcal{P}$ is a convex polygon, then the extreme points of $\mathcal{P}$ are its vertices and no other points.*

Theorem 8.4 (Vertex Criterion for Convexity). *A polygon $\mathcal{P}$ is convex if and only if for every edge of $\mathcal{P}$, the vertices of $\mathcal{P}$ that are not on that edge all lie on the same side of the line containing the edge.*

Corollary 8.5. *Every triangle is a convex polygon.*

Theorem 8.6 (Angle Criterion for Convexity). *A polygon $\mathcal{P}$ is convex if and only if for each vertex A_i of $\mathcal{P}$, all the vertices of $\mathcal{P}$ are contained in the interior of $\angle A_i$ except A_i itself and the two vertices consecutive with it.*

Theorem 8.7 (Semiparallel Criterion for Convexity). *A polygon is convex if and only if all pairs of nonadjacent edges are semiparallel.*

Lemma 8.8. *If $\mathcal{P}$ is a convex polygon and $\overline{BC}$ is a chord of $\mathcal{P}$, then $\operatorname{Int}\overline{BC}$ is disjoint from $\mathcal{P}$.*

Theorem 8.9 (Polygon Splitting Theorem). *If $\mathcal{P}$ is a convex polygon and $\overline{BC}$ is a chord of $\mathcal{P}$, then the two subpolygons cut off by $\overline{BC}$ are both convex polygons.*

Lemma 8.10. *Suppose $\mathcal{P}$ is a convex polygon and Q is a point in the plane.*

(*a*) *$Q \in \operatorname{Int}\mathcal{P}$ if and only if Q lies on the $\mathcal{P}$-side of every edge line.*

(*b*) *$Q \in \operatorname{Reg}\mathcal{P}$ if and only if for each edge of $\mathcal{P}$, Q lies in the closed half-plane determined by that edge and the vertices not on that edge.*

Lemma 8.11. *Suppose $\mathcal{P}$ is a convex polygon, A is a point in $\operatorname{Int}\mathcal{P}$, and B is a point in $\operatorname{Ext}\mathcal{P}$. Then $\overline{AB}$ intersects $\mathcal{P}$.*

Theorem 8.12. *If $\mathcal{P}$ is a convex polygon, then both $\operatorname{Int}\mathcal{P}$ and $\operatorname{Reg}\mathcal{P}$ are convex sets.*

Theorem 8.13 (Jordan Polygon Theorem). *Suppose $\mathcal{P}$ is a polygon.*

(*a*) *Every point not on $\mathcal{P}$ is in either $\operatorname{Int}\mathcal{P}$ or $\operatorname{Ext}\mathcal{P}$, but not both.*

(*b*) *If A and B are two points such that $A \in \operatorname{Int}\mathcal{P}$ and $B \in \operatorname{Ext}\mathcal{P}$, then every polygonal path from A to B intersects $\mathcal{P}$.*

(*c*) *If A and B are two points that both lie in $\operatorname{Int}\mathcal{P}$ or both lie in $\operatorname{Ext}\mathcal{P}$, then there is a polygonal path from A to B that lies entirely in $\operatorname{Int}\mathcal{P}$ or $\operatorname{Ext}\mathcal{P}$, respectively.*

Lemma 8.14. *If $\angle CQD$ is a proper angle and $\overline{AB}$ is a segment such that A and B are not on $\angle CQD$ and Q is not on $\overline{AB}$, then the number of intersections of $\overline{AB}$ with $\angle CQD$ is exactly one when one endpoint of $\overline{AB}$ is in the interior of the angle and the other is in the exterior; and otherwise it is zero or two.*

Theorem 8.15. *Suppose $\mathcal{P}$ is a convex polygon and Q is any point not on $\mathcal{P}$. Then Q has odd parity if and only if Q is in the interior of each of the angles of $\mathcal{P}$.*

Corollary 8.16. *Suppose $\mathcal{P}$ is a convex polygon and $Q \in \operatorname{Int}\mathcal{P}$. Then every ray starting at Q intersects $\mathcal{P}$ exactly once.*

Lemma 8.17. *If $\mathcal{P}$ is a convex polygon, then every vertex of $\mathcal{P}$ is a convex vertex.*

Lemma 8.18. *If $\mathcal{P}$ is any polygon, then every vertex of $\mathcal{P}$ is either convex or concave.*

Theorem 8.19 (Characterizations of Convex Polygons). *If $\mathcal{P}$ is a polygon, the following are equivalent:*

(*a*) *$\mathcal{P}$ is a convex polygon.*

(b) Int $\mathcal{P}$ *is a convex set.*
(c) Reg $\mathcal{P}$ *is a convex set.*
(d) *All vertices of* $\mathcal{P}$ *are convex.*

Theorem 9.1. *Every rectangle is a parallelogram.*

Lemma 9.2. *In a convex quadrilateral, each pair of opposite vertices lies on opposite sides of the line through the other two vertices.*

Lemma 9.3. *Suppose $ABCD$ is a convex quadrilateral. Then $m\angle BAD = m\angle BAC + m\angle CAD$, with similar statements for the angles at the other vertices.*

Theorem 9.4 (Diagonal Criterion for Convex Quadrilaterals).

(a) *If the diagonals of a quadrilateral intersect, then the quadrilateral is convex.*

(b) *If a quadrilateral is convex, then its diagonals intersect at a point that is in the interiors of both diagonals and of the quadrilateral.*

Theorem 9.5. *If a quadrilateral has at least one pair of semiparallel sides, it is convex.*

Corollary 9.6. *Every trapezoid is a convex quadrilateral.*

Corollary 9.7. *Every parallelogram is a convex quadrilateral.*

Corollary 9.8. *Every rectangle is a convex quadrilateral.*

Lemma 9.9 (Cross Lemma). *Suppose $\overline{AC}$ and $\overline{BD}$ are noncollinear segments that have an interior point in common. Then $ABCD$ is a convex quadrilateral.*

Lemma 9.10 (Trapezoid Lemma). *Suppose A, B, C, and D are four distinct points such that $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \cap \overline{BC} = \varnothing$. Then $ABCD$ is a trapezoid.*

Lemma 9.11 (Parallelogram Lemma). *Suppose A, B, C, and D are four distinct points such that $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$. Then $ABCD$ is a parallelogram.*

Theorem 9.12 (SASAS Congruence). *Suppose $ABCD$ and $EFGH$ are convex quadrilaterals such that $\overline{AB} \cong \overline{EF}$, $\overline{BC} \cong \overline{FG}$, $\overline{CD} \cong \overline{GH}$, $\angle B \cong \angle F$, and $\angle C \cong \angle G$. Then $ABCD \cong EFGH$.*

Theorem 9.13 (AASAS Congruence). *Suppose $ABCD$ and $EFGH$ are convex quadrilaterals such that $\angle A \cong \angle E$, $\angle B \cong \angle F$, $\angle C \cong \angle G$, $\overline{BC} \cong \overline{FG}$, and $\overline{CD} \cong \overline{GH}$. Then $ABCD \cong EFGH$.*

Theorem 9.14 (Quadrilateral Copying Theorem). *Suppose $ABCD$ is a convex quadrilateral, and $\overline{EF}$ is a segment congruent to $\overline{AB}$. On either side of $\overleftrightarrow{EF}$, there are distinct points G and H such that $EFGH \cong ABCD$.*

Theorem 9.15. *A convex quadrilateral with two pairs of congruent opposite angles is a parallelogram.*

Corollary 9.16. *Every equiangular quadrilateral is a parallelogram.*

Theorem 9.17. *A quadrilateral with two pairs of congruent opposite sides is a parallelogram.*

Corollary 9.18. *Every rhombus is a parallelogram.*

Theorem 9.19. *Suppose $ABCD$ is a quadrilateral.*

(a) *If its diagonals bisect each other, then $ABCD$ is a parallelogram.*

(b) *If its diagonals are congruent and bisect each other, then $ABCD$ is equiangular.*

(c) *If its diagonals are perpendicular bisectors of each other, then $ABCD$ is a rhombus.*

(d) *If its diagonals are congruent and are perpendicular bisectors of each other, then $ABCD$ is a regular quadrilateral.*

Corollary 9.20. *There exists a regular quadrilateral.*

Postulates of Euclidean Geometry

Postulates 1–9 of Neutral Geometry.

Postulate 10E (Euclidean Parallel Postulate). *For each line ℓ and each point A that does not lie on ℓ, there is a unique line that contains A and is parallel to ℓ.*

Postulate 11E (The Euclidean Area Postulate). *There exists a unique area function α with the property that $\alpha(\mathcal{R}) = 1$ whenever $\mathcal{R}$ is a square region with sides of length 1.*

Theorems of Euclidean Geometry

All of the theorems of neutral geometry.

Theorem 10.1 (Converse to the Alternate Interior Angles Theorem). *If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.*

Corollary 10.2 (Converse to the Corresponding Angles Theorem). *If two parallel lines are cut by a transversal, then all four pairs of corresponding angles are congruent.*

Corollary 10.3 (Converse to the Consecutive Interior Angles Theorem). *If two parallel lines are cut by a transversal, then both pairs of consecutive interior angles are supplementary.*

Lemma 10.4 (Proclus's Lemma). *Suppose ℓ and ℓ' are parallel lines. If t is a line that is distinct from ℓ but intersects ℓ, then t also intersects ℓ'.*

Theorem 10.5. *Suppose ℓ and ℓ' are parallel lines. Then any line that is perpendicular to one of them is perpendicular to both.*

Corollary 10.6. *Suppose ℓ and ℓ' are parallel lines, and m and m' are distinct lines such that $m \perp \ell$ and $m' \perp \ell'$. Then $m \parallel m'$.*

Corollary 10.7 (Converse to the Common Perpendiculars Theorem). *If two lines are parallel, then they have a common perpendicular.*

Theorem 10.8 (Converse to the Equidistance Theorem). *If two lines are parallel, then each one is equidistant from the other.*

Corollary 10.9 (Symmetry of Equidistant Lines). *If ℓ and m are two distinct lines, then ℓ is equidistant from m if and only if m is equidistant from ℓ.*

Theorem 10.10 (Transitivity of Parallelism). *If ℓ, m, and n are distinct lines such that $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.*

Theorem 10.11 (Angle-Sum Theorem for Triangles). *Every triangle has angle sum equal to $180°$.*

Corollary 10.12. *In any triangle, the measure of each exterior angle is equal to the sum of the measures of the two remote interior angles.*

Theorem 10.13 (60-60-60 Theorem). *A triangle has all of its interior angle measures equal to $60°$ if and only if it is equilateral.*

Theorem 10.14 (30-60-90 Theorem). *A triangle has interior angle measures $30°$, $60°$, and $90°$ if and only if it is a right triangle in which the hypotenuse is twice as long as one of the legs.*

Theorem 10.15 (45-45-90 Theorem). *A triangle has interior angle measures $45°$, $45°$, and $90°$ if and only if it is an isosceles right triangle.*

Theorem 10.16 (Euclid's Fifth Postulate). *If ℓ and ℓ' are two lines cut by a transversal t in such a way that the measures of two consecutive interior angles add up to less than $180°$, then ℓ and ℓ' intersect on the same side of t as those two angles.*

Theorem 10.17 (AAA Construction Theorem). *Suppose $\overline{AB}$ is a segment, and α, β, and γ are three positive real numbers whose sum is 180. On each side of $\overleftrightarrow{AB}$, there is a point C such that $\triangle ABC$ has the following angle measures: $m\angle A = \alpha°$, $m\angle B = \beta°$, and $m\angle C = \gamma°$.*

Corollary 10.18 (Equilateral Triangle Construction Theorem). *If $\overline{AB}$ is any segment, then on each side of $\overleftrightarrow{AB}$ there is a point C such that $\triangle ABC$ is equilateral.*

Theorem 10.19 (Angle-Sum Theorem for Convex Polygons). *In a convex polygon with n sides, the angle sum is equal to $(n-2)\times 180°$.*

Corollary 10.20. *In a regular n-gon, the measure of each angle is $\frac{n-2}{n} \times 180°$.*

Corollary 10.21 (Exterior Angle Sum for a Convex Polygon). *In any convex polygon, the sum of the measures of the exterior angles (one at each vertex) is $360°$.*

Theorem 10.22 (Angle-Sum Theorem for General Polygons). *If $\mathscr{P}$ is any polygon with n sides, the sum of its interior angle measures is $(n-2)\times 180°$.*

Theorem 10.23 (Angle-Sum Theorem for Quadrilaterals). *Every convex quadrilateral has an angle sum of $360°$.*

Corollary 10.24. *A quadrilateral is equiangular if and only if it is a rectangle, and it is a regular quadrilateral if and only if it is a square.*

Theorem 10.25. *Every parallelogram has the following properties.*

(*a*) *Each diagonal cuts it into a pair of congruent triangles.*
(*b*) *Both pairs of opposite sides are congruent.*
(*c*) *Both pairs of opposite angles are congruent.*
(*d*) *Its diagonals bisect each other.*

Theorem 10.26. *If a quadrilateral has a pair of congruent and parallel opposite sides, then it is a parallelogram.*

Theorem 10.27. *If a quadrilateral has a pair of congruent opposite sides that are both perpendicular to a third side, then it is a rectangle.*

Theorem 10.28 (Rectangle Construction). *Suppose a and b are positive real numbers, and $\overline{AB}$ is a segment of length a. On either side of $\overleftrightarrow{AB}$, there exist points C and D such that $ABCD$ is a rectangle with $AB = CD = a$ and $AD = BC = b$.*

Corollary 10.29 (Square Construction). *If $\overline{AB}$ is any segment, then on each side of $\overleftrightarrow{AB}$ there are points C and D such that $ABCD$ is a square.*

Theorem 10.30 (Midsegment Theorem). *Any midsegment of a triangle is parallel to the third side and half as long.*

Lemma 11.1 (Convex Decomposition Lemma). *Suppose $\mathcal{P}$ is a convex polygon, and $\overline{BC}$ is a chord of $\mathcal{P}$. Then the two convex polygons $\mathcal{P}_1$ and $\mathcal{P}_2$ described in the polygon splitting theorem (Theorem) form an admissible decomposition of $\mathcal{P}$, and therefore $\alpha(\mathcal{P}) = \alpha(\mathcal{P}_1) + \alpha(\mathcal{P}_2)$.*

Lemma 11.2 (Pizza Lemma). *Suppose $\mathcal{P}$ is a convex polygon, O is a point in $\operatorname{Int}\mathcal{P}$, and $\{B_1, \dots, B_m\}$ are distinct points on $\mathcal{P}$, ordered in such a way that for each $i = 1, \dots, m$, the angle $\angle B_i O B_{i+1}$ is proper and contains none of the B_j's in its interior (where we interpret B_{m+1} to mean B_1). For each $i = 1, \dots, m$, let $\mathcal{P}_i$ denote the following set: $\mathcal{P}_i = \overline{OB_i} \cup \overline{OB_{i+1}} \cup \left(\mathcal{P} \cap \operatorname{Int}\angle B_i O B_{i+1}\right)$. Then each $\mathcal{P}_i$ is a convex polygon, and $\alpha(\mathcal{P}) = \alpha(\mathcal{P}_1) + \cdots + \alpha(\mathcal{P}_m)$.*

Lemma 11.3 (Parallelogram Decomposition Lemma). *Suppose $ABCD$ is a parallelogram, and E, F, G, H are interior points on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, such that $\overline{AH} \cong \overline{BF}$ and $\overline{AE} \cong \overline{DG}$. Then there is a point X where $\overline{HF}$ intersects $\overline{EG}$, and $AEXH$, $EBFX$, $HXGD$, and $XFCG$ are parallelograms such that $\alpha(ABCD) = \alpha(AEXH) + \alpha(EBFX) + \alpha(HXGD) + \alpha(XFCG)$.*

Theorem 11.7 (Area of a Square). *The area of a square of side length x is x^2.*

Theorem 11.8 (Area of a Rectangle). *The area of a rectangle is the product of the lengths of any two adjacent sides.*

Lemma 11.9 (Area of a Right Triangle). *The area of a right triangle is one-half of the product of the lengths of its legs.*

Theorem 11.10 (Area of a Triangle). *The area of a triangle is equal to one-half the length of any base multiplied by the corresponding height.*

Corollary 11.11 (Triangle Sliding Theorem). *Suppose $\triangle ABC$ and $\triangle A'BC$ are triangles with a common side $\overline{BC}$, such that A and A' both lie on a line parallel to $\overleftrightarrow{BC}$. Then $\alpha(\triangle ABC) = \alpha(\triangle A'BC)$.*

Corollary 11.12 (Triangle Area Proportion Theorem). *Suppose $\triangle ABC$ and $\triangle AB'C'$ are triangles with a common vertex A, such that the points B, C, B', C' are collinear. Then*

$$\frac{\alpha(\triangle ABC)}{\alpha(\triangle AB'C')} = \frac{BC}{B'C'}.$$

Theorem 11.13 (Area of a Trapezoid). *The area of a trapezoid is the average of the lengths of the bases multiplied by the height.*

Corollary 11.14 (Area of a Parallelogram). *The area of a parallelogram is the length of any base multiplied by the corresponding height.*

Theorem 12.1 (Transitive Property of Similarity). *Two polygons that are both similar to a third polygon are similar to each other.*

Theorem 12.2 (The Side-Splitter Theorem). *Suppose $\triangle ABC$ is a triangle, and ℓ is a line parallel to $\overleftrightarrow{BC}$ that intersects $\overline{AB}$ at an interior point D. Then ℓ also intersects $\overline{AC}$ at an interior point E, and the following proportions hold:*

$$\frac{AD}{AB} = \frac{AE}{AC} \qquad \text{and} \qquad \frac{AD}{DB} = \frac{AE}{EC}.$$

Theorem 12.3 (AA Similarity Theorem). *If there is a correspondence between the vertices of two triangles such that two pairs of corresponding angles are congruent, then the triangles are similar under that correspondence.*

Theorem 12.4 (Similar Triangle Construction Theorem). *If $\triangle ABC$ is a triangle and $\overline{DE}$ is any segment, then on each side of $\overleftrightarrow{DE}$, there is a point F such that $\triangle ABC \sim \triangle DEF$.*

Theorem 12.5 (SSS Similarity Theorem). *If $\triangle ABC$ and $\triangle DEF$ are triangles such that $AB/DE = AC/DF = BC/EF$, then $\triangle ABC \sim \triangle DEF$.*

Theorem 12.6 (SAS Similarity Theorem). *If $\triangle ABC$ and $\triangle DEF$ are triangles such that $\angle A \cong \angle D$ and $AB/DE = AC/DF$, then $\triangle ABC \sim \triangle DEF$.*

Theorem 12.7 (Two Transversals Theorem). *Suppose ℓ and ℓ' are parallel lines, and m and n are two distinct transversals to ℓ and ℓ' meeting at a point X that is not on either ℓ or ℓ'. Let M and N be the points where m and n, respectively, meet ℓ; and let M' and N' be the points where they meet ℓ'. Then $\triangle XMN \sim \triangle XM'N'$.*

Theorem 12.8 (Converse to the Side-Splitter Theorem). *Suppose $\triangle ABC$ is a triangle, and D and E are interior points on $\overline{AB}$ and $\overline{AC}$, respectively, such that $AD/AB = AE/AC$. Then $\overleftrightarrow{DE}$ is parallel to $\overleftrightarrow{BC}$.*

Theorem 12.9 (Angle Bisector Proportion Theorem). *Suppose $\triangle ABC$ is a triangle and D is the point where the bisector of $\angle BAC$ meets $\overline{BC}$. Then $BD/DC = AB/AC$.*

Theorem 12.10 (Parallel Projection Theorem). *Suppose ℓ, m, n, t, and t' are distinct lines such that $\ell \parallel m \parallel n$; t intersects ℓ, m, and n at A, B, and C, respectively; and t' intersects the same three lines at A', B', and C', respectively. Then*

(*a*) $\dfrac{AB}{BC} = \dfrac{A'B'}{B'C'}$.

(*b*) $A * B * C$ *if and only if* $A' * B' * C'$.

Theorem 12.11 (Menelaus's Theorem). *Let $\triangle ABC$ be a triangle. Suppose D, E, F are points different from A, B, C and lying on $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$, respectively, such that either two of the points lie on $\triangle ABC$ or none of them do. Then D, E, and F are collinear if and only if*

$$\left(\frac{AD}{DB}\right)\left(\frac{BE}{EC}\right)\left(\frac{CF}{FA}\right) = 1.$$

Theorem 12.12 (Ceva's Theorem). *Suppose $\triangle ABC$ is a triangle, and D, E, F are points in the interiors of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Then the three cevians $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ are concurrent if and only if*

$$\left(\frac{AD}{DB}\right)\left(\frac{BE}{EC}\right)\left(\frac{CF}{FA}\right) = 1.$$

Theorem 12.13 (Median Concurrence Theorem). *The medians of a triangle are concurrent, and the distance from the point of intersection to each vertex is twice the distance to the midpoint of the opposite side.*

Theorem 12.14 (Golden Rectangle Theorem). *A rectangle is a golden rectangle if and only if the ratio of the longer side length to the shorter one is equal to the following number, called the* ***golden ratio****:*

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

Theorem 12.15 (36-72-72 Theorem). *A triangle is a golden triangle if and only if it has angle measures 36°, 72°, and 72°.*

Theorem 12.16 (Perimeter Scaling Theorem). *If two polygons are similar, then the ratio of their perimeters is the same as the ratio of their corresponding side lengths.*

Theorem 12.17 (Height Scaling Theorem). *If two triangles are similar, their corresponding heights have the same ratio as their corresponding side lengths.*

Theorem 12.18 (Diagonal Scaling Theorem). *If two convex quadrilaterals are similar, the lengths of their corresponding diagonals have the same ratio as their corresponding side lengths.*

Theorem 12.19 (Triangle Area Scaling Theorem). *If two triangles are similar, then the ratio of their areas is the square of the ratio of their corresponding side lengths; that is, if $\triangle ABC \sim \triangle DEF$ and $AB = r \cdot DE$, then $\alpha(\triangle ABC) = r^2 \cdot \alpha(\triangle DEF)$.*

Theorem 12.20 (Quadrilateral Area Scaling Theorem). *If two convex quadrilaterals are similar, then the ratio of their areas is the square of the ratio of their corresponding side lengths.*

Theorem 13.1 (The Pythagorean Theorem). *Suppose $\triangle ABC$ is a right triangle with right angle at C, and let a, b, and c denote the lengths of the sides opposite A, B, and C, respectively. Then $a^2 + b^2 = c^2$.*

Theorem 13.2 (Converse to the Pythagorean Theorem). *Suppose $\triangle ABC$ is a triangle with side lengths a, b, and c. If $a^2 + b^2 = c^2$, then $\triangle ABC$ is a right triangle, and its hypotenuse is the side of length c.*

Theorem 13.3 (Side Lengths of 30-60-90 Triangles). *In a triangle with angle measures 30°, 60°, and 90°, the longer leg is $\sqrt{3}$ times as long as the shorter leg, and the hypotenuse is twice as long as the shorter leg.*

Theorem 13.4 (Side Lengths of 45-45-90 Triangles). *In a triangle with angle measures 45°, 45°, and 90°, the legs are congruent, and the hypotenuse is $\sqrt{2}$ times as long as either leg.*

Theorem 13.5 (Diagonal of a Square). *In a square, each diagonal is $\sqrt{2}$ times as long as each side.*

Theorem 13.6 (SSS Existence Theorem). *Suppose a, b, and c are positive real numbers such that each one is strictly less than the sum of the other two. Then there exists a triangle with side lengths a, b, and c.*

Corollary 13.7 (SSS Construction Theorem). *Suppose a, b, and c are positive real numbers such that each one is strictly less than the sum of the other two, and $\overline{AB}$ is a segment of length c. Then on either side of $\overleftrightarrow{AB}$, there exists a unique point C such that $\triangle ABC$ has side lengths a, b, and c opposite vertices A, B, and C, respectively.*

Theorem 13.8 (Right Triangle Similarity Theorem). *The altitude to the hypotenuse of a right triangle cuts it into two triangles that are similar to each other and to the original triangle.*

Theorem 13.9 (Right Triangle Proportion Theorem). *In every right triangle, the following proportions hold:*

(*a*) *The altitude to the hypotenuse is the geometric mean of the projections of the two legs onto the hypotenuse.*

(*b*) *Each leg is the geometric mean of its projection onto the hypotenuse and the whole hypotenuse.*

Theorem 13.10. *Suppose* $\theta \in [0,180]$. *Then*

$$\begin{aligned} 0 < \sin\theta < 1 \quad &\textit{if and only if} \quad 0^\circ < \theta < 90^\circ \textit{ or } 90^\circ < \theta < 180^\circ; \\ 0 < \cos\theta < 1 \quad &\textit{if and only if} \quad 0^\circ < \theta < 90^\circ; \\ -1 < \cos\theta < 0 \quad &\textit{if and only if} \quad 90^\circ < \theta < 180^\circ. \end{aligned}$$

Theorem 13.11. *The cosine function is injective.*

Theorem 13.12 (The Pythagorean Identity). *If* θ *is any real number in the interval* $[0,180]$, *then* $\sin^2\theta + \cos^2\theta = 1$.

Theorem 13.13 (Law of Cosines). *Let* $\triangle ABC$ *be any triangle, and let* a, b, *and* c *denote the lengths of the sides opposite* A, B, *and* C, *respectively. Then* $a^2 + b^2 = c^2 + 2ab\cos\angle C$.

Theorem 13.14 (Law of Sines). *Let* $\triangle ABC$ *be any triangle, and let* a, b, *and* c *denote the lengths of the sides opposite* A, B, *and* C, *respectively. Then*

$$\frac{\sin\angle A}{a} = \frac{\sin\angle B}{b} = \frac{\sin\angle C}{c}.$$

Theorem 13.15 (Sum Formulas). *Suppose* α *and* β *are real numbers such that* α, β, *and* $\alpha+\beta$ *are all strictly between* 0° *and* 90°. *Then*

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta, \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta. \end{aligned}$$

Corollary 13.16 (Double Angle Formulas). *Suppose* α *is a real number strictly between* 0° *and* 45°. *Then*

$$\begin{aligned} \sin 2\alpha &= 2\sin\alpha\cos\alpha, \\ \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha. \end{aligned}$$

Corollary 13.17 (Triple Angle Formulas). *Suppose* α *is a real number strictly between* 0° *and* 30°. *Then*

$$\begin{aligned} \sin 3\alpha &= 3\sin\alpha - 4\sin^3\alpha, \\ \cos 3\alpha &= 4\cos^3\alpha - 3\cos\alpha. \end{aligned}$$

Theorem 13.18 (Heron's Formula). *Let* $\triangle ABC$ *be a triangle, and let* a,b,c *denote the lengths of the sides opposite* A, B, *and* C, *respectively. Then* $\alpha(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}$, *where* $s = (a+b+c)/2$ *(called the* ***semiperimeter of*** $\boldsymbol{\triangle ABC}$*).*

Theorem 13.19. *The postulates for Euclidean geometry are categorical.*

Lemma 14.1. *If* $\mathcal{C}$ *is a circle of radius* r, *then every diameter of* $\mathcal{C}$ *has length* $2r$, *and the center of* $\mathcal{C}$ *is the midpoint of each diameter.*

Lemma 14.2. *Two concentric circles that have a point in common are equal.*

Theorem 14.3. *No circle contains three distinct collinear points.*

Theorem 14.4 (Properties of Secant Lines). *Suppose* $\mathcal{C}$ *is a circle and* ℓ *is a secant line that intersects* $\mathcal{C}$ *at* A *and* B. *Then every interior point of the chord* $\overline{AB}$ *is in the interior of* $\mathcal{C}$, *and every point of* ℓ *that is not in* $\overline{AB}$ *is in the exterior of* $\mathcal{C}$.

Theorem 14.5 (Properties of Chords). *Suppose* $\mathcal{C}$ *is a circle and* $\overline{AB}$ *is a chord of* $\mathcal{C}$.

(*a*) *The perpendicular bisector of* $\overline{AB}$ *passes through the center of* $\mathcal{C}$.

(*b*) *If* $\overline{AB}$ *is not a diameter, a radius of* $\mathcal{C}$ *is perpendicular to* $\overline{AB}$ *if and only if it bisects* $\overline{AB}$.

Theorem 14.6 (Line-Circle Theorem). *Suppose* $\mathcal{C}$ *is a circle and* ℓ *is a line that contains a point in the interior of* $\mathcal{C}$. *Then* ℓ *is a secant line for* $\mathcal{C}$, *and thus there are exactly two points where* ℓ *intersects* $\mathcal{C}$.

Theorem 14.7 (Tangent Line Theorem). *Suppose* $\mathcal{C}$ *is a circle, and* ℓ *is a line that intersects* $\mathcal{C}$ *at a point* P. *Then* ℓ *is tangent to* $\mathcal{C}$ *if and only if* ℓ *is perpendicular to the radius through* P.

Corollary 14.8. *If* $\mathcal{C}$ *is a circle and* $A \in \mathcal{C}$, *there is a unique line tangent to* $\mathcal{C}$ *at* A.

Theorem 14.9 (Properties of Tangent Lines). *If $\mathcal{C}$ is a circle and ℓ is a line that is tangent to $\mathcal{C}$ at P, then every point of ℓ except P lies in the exterior of $\mathcal{C}$, and every point of $\mathcal{C}$ except P lies on the same side of ℓ as the center of $\mathcal{C}$.*

Theorem 14.10 (Two Circles Theorem). *Suppose $\mathcal{C}$ and $\mathcal{D}$ are two circles. If either of the following conditions is satisfied, then there exist exactly two points where the circles intersect, one on each side of the line containing their centers.*

- (a) *The following inequalities all hold: $d < r + s$, $s < r + d$, and $r < d + s$, where r and s are the radii of $\mathcal{C}$ and $\mathcal{D}$, respectively, and d is the distance between their centers.*
- (b) *$\mathcal{D}$ contains a point in the interior of $\mathcal{C}$ and a point in the exterior of $\mathcal{C}$.*

Theorem 14.11 (Tangent Circles Theorem). *Suppose $\mathcal{C}(O,r)$ and $\mathcal{C}(Q,s)$ are tangent to each other at A. Then O, Q, and A are distinct and collinear, and $\mathcal{C}(O,r)$ and $\mathcal{C}(Q,s)$ have a common tangent line at A.*

Lemma 14.12. *Suppose $\mathcal{C}$ is a circle and AB is a minor arc of $\mathcal{C}$. If X is any point on $\mathcal{C}$, then X lies on AB if and only if the radius $\overline{OX}$ meets the chord $\overline{AB}$.*

Theorem 14.13. *Suppose $\mathcal{C}$ is a circle with center O, and A, B are two distinct points on $\mathcal{C}$. Then the two arcs bounded by A and B are the intersections with $\mathcal{C}$ of the closed half-planes determined by $\overleftrightarrow{AB}$.*

Theorem 14.14 (Consistency of Endpoints of Arcs). *Suppose A, B, C, D are points on a circle $\mathcal{C}$, with $A \neq B$ and $C \neq D$. If $AB = CD$, then $\{A, B\} = \{C, D\}$.*

Lemma 14.15. *Any two conjugate arcs have measures adding up to $360°$.*

Theorem 14.16 (Thales's Theorem). *Any angle inscribed in a semicircle is a right angle.*

Theorem 14.17 (Converse to Thales's Theorem). *The hypotenuse of a right triangle is a diameter of a circle that contains all three vertices.*

Theorem 14.18 (Existence of Tangent Lines Through an Exterior Point). *Let $\mathcal{C}$ be a circle, and let A be a point in the exterior of $\mathcal{C}$. Then there are exactly two distinct tangent lines to $\mathcal{C}$ containing A. The two points of tangency X and Y are equidistant from A, and the center of $\mathcal{C}$ lies on the bisector of $\angle XAY$.*

Theorem 14.19 (Inscribed Angle Theorem). *The measure of a proper angle inscribed in a circle is one-half the measure of its intercepted arc.*

Corollary 14.20 (Arc Addition Theorem). *Suppose A, B, and C are three distinct points on a circle $\mathcal{C}$, and AB and BC are arcs that intersect only at B. Then $mABC = mAB + mBC$.*

Corollary 14.21 (Intersecting Chords Theorem). *Suppose $\overline{AB}$ and $\overline{CD}$ are two distinct chords of a circle $\mathcal{C}$ that intersect at a point $P \in \operatorname{Int}\mathcal{C}$. Then $(PA)(PB) = (PC)(PD)$.*

Corollary 14.22 (Intersecting Secants Theorem). *Suppose two distinct secant lines of a circle $\mathcal{C}$ intersect at a point P exterior to $\mathcal{C}$. Let A, B be the points where one of the secants meets $\mathcal{C}$, and C, D be the points where the other one does. Then $(PA)(PB) = (PC)(PD)$.*

Theorem 14.23. *Every polygon inscribed in a circle is convex.*

Theorem 14.24 (Circumcircle Theorem). *A polygon $\mathcal{P}$ is cyclic if and only if the perpendicular bisectors of all of its edges are concurrent. If this is the case, the point O where these perpendicular bisectors intersect is the unique circumcenter for $\mathcal{P}$, and the circle with center O and radius equal to the distance from O to any vertex is the unique circumcircle.*

Theorem 14.25 (Cyclic Triangle Theorem). *Every triangle is cyclic.*

Corollary 14.26 (Perpendicular Bisector Concurrence Theorem). *In any triangle, the perpendicular bisectors of all three sides are concurrent.*

Corollary 14.27. *Given three noncollinear points, there is a unique circle that contains all of them.*

Theorem 14.28 (Cyclic Quadrilateral Theorem). *A quadrilateral $ABCD$ is cyclic if and only if it is convex and both pairs of opposite angles are supplementary: $m\angle A + m\angle C = 180°$ and $m\angle B + m\angle D = 180°$.*

Lemma 14.29. *If $\mathcal{C}$ is an incircle for a polygon $\mathcal{P}$, then the point of tangency for each edge is an interior point of the edge.*

Theorem 14.31. *Every tangential polygon is convex.*

Corollary 14.32. *If $\mathcal{P}$ is a tangential polygon, then the interior of its incircle is contained in the interior of $\mathcal{P}$.*

Theorem 14.33 (Incircle Theorem). *A polygon $\mathcal{P}$ is tangential if and only if it is convex and the bisectors of all of its angles are concurrent. If this is the case, the point O where these bisectors intersect is the unique incenter for $\mathcal{P}$, and the circle with center O and radius equal to the distance from O to any edge line is the unique incircle.*

Theorem 14.34 (Tangential Triangle Theorem). *Every triangle is tangential.*

Corollary 14.35 (Angle Bisector Concurrence Theorem). *In any triangle, the bisectors of the three angles are concurrent.*

Theorem 14.36 (Altitude Concurrence Theorem). *In any triangle, the lines containing the three altitudes are concurrent.*

Lemma 14.37. *Every kite is a tangential quadrilateral.*

Theorem 14.38 (Tangential Quadrilateral Theorem). *A quadrilateral $ABCD$ is tangential if and only if it is convex and both pairs of opposite side lengths have the same sum: $AB + CD = AD + BC$.*

Theorem 14.39. *Every regular polygon is both cyclic and tangential, and its incenter is equal to its circumcenter.*

Theorem 14.40 (Central Angles of a Regular Polygon). *Every central angle of a regular n-gon has measure $360^\circ/n$.*

Lemma 14.41. *Every equilateral polygon inscribed in a circle is regular.*

Theorem 14.42. *Suppose $\mathcal{C}$ is a circle, n is an integer greater than or equal to 3, and P is any point on $\mathcal{C}$. Then there is a regular n-gon inscribed in $\mathcal{C}$ that has P as one of its vertices.*

Theorem 14.43. *Suppose $\mathcal{C}$ is a circle, n is an integer greater than or equal to 3, and P is any point on $\mathcal{C}$. Then there is a regular n-gon circumscribed about $\mathcal{C}$ that has one of its vertices on $\overrightarrow{OP}$.*

Theorem 14.44. *Let $\mathcal{C}$ be a circle. Given an integer $n \geq 3$, any two regular n-gons inscribed in $\mathcal{C}$ are congruent to each other, as are any two regular n-gons circumscribed about $\mathcal{C}$.*

Lemma 15.1. *Suppose $\mathcal{C}$ is a circle. Given any polygon inscribed in $\mathcal{C}$, there is another inscribed polygon with larger perimeter and larger area; and given any polygon circumscribed about $\mathcal{C}$, there is another circumscribed polygon with smaller perimeter and smaller area.*

Lemma 15.2. *Suppose $\mathcal{C}$ is a circle with center O, and $\mathcal{P}$ is a polygon inscribed in $\mathcal{C}$ such that $O \notin \operatorname{Int}\mathcal{P}$. Then there is another polygon $\mathcal{P}'$ inscribed in $\mathcal{C}$ that has larger area and perimeter than $\mathcal{P}$, and that satisfies $O \in \operatorname{Int}\mathcal{P}'$.*

Theorem 15.3. *The perimeter of every polygon circumscribed about a circle is strictly larger than the perimeter of every polygon inscribed in the same circle.*

Corollary 15.4. *Let $\mathcal{C}$ be a circle. If $\mathcal{P}$ is any polygon inscribed in $\mathcal{C}$ and $\mathcal{Q}$ is any polygon circumscribed about $\mathcal{C}$, then $\operatorname{perim}(\mathcal{P}) < \operatorname{circum}(\mathcal{C}) < \operatorname{perim}(\mathcal{Q})$.*

Theorem 15.5. *Suppose $\mathcal{C}$ is a circle of radius r, $\mathcal{P}$ is a regular n-gon inscribed in $\mathcal{C}$, and $\mathcal{Q}$ is a regular n-gon circumscribed about $\mathcal{C}$, with $n \geq 12$. Then $\operatorname{perim}(\mathcal{Q}) - \operatorname{perim}(\mathcal{P}) < 800r/n^2$.*

Corollary 15.6. *Let $\mathcal{C}$ be a circle, and for each n let $\mathcal{P}_n$ be a regular n-gon inscribed in $\mathcal{C}$. Then $\operatorname{circum}(\mathcal{C}) = \lim_{n\to\infty} \operatorname{perim}(\mathcal{P}_n)$.*

Theorem 15.7 (Circumference Scaling Theorem). *For any two circles, the ratio of their circumferences is the same as the ratio of their radii.*

Corollary 15.8 (Circumference of a Circle). *The circumference of a circle of radius r is $2\pi r$.*

Corollary 15.9. *Let $\mathcal{C}$ be a unit circle. If $\mathcal{P}$ is any polygon inscribed in $\mathcal{C}$ and $\mathcal{Q}$ is any polygon circumscribed about $\mathcal{C}$, then $\frac{1}{2}\operatorname{perim}(\mathcal{P}) \leq \pi \leq \frac{1}{2}\operatorname{perim}(\mathcal{Q})$.*

Theorem 15.10. $3 < \pi < 4$.

Theorem 15.11. *Let $\mathcal{C}$ be a circle, let $\mathcal{P}$ be a polygon inscribed in $\mathcal{C}$, and let $\mathcal{Q}$ be a polygon circumscribed about $\mathcal{C}$. Then $\alpha(\mathcal{Q}) > \alpha(\mathcal{P})$.*

Theorem 15.12. *Suppose $\mathcal{C}$ is a circle of radius r, $\mathcal{P}$ is a regular n-gon inscribed in $\mathcal{C}$, and $\mathcal{Q}$ is a regular n-gon circumscribed about $\mathcal{C}$, with $n \geq 12$. Then $\alpha(\mathcal{Q}) - \alpha(\mathcal{P}) < 800r^2/n^2$.*

Corollary 15.13. *Let $\mathcal{C}$ be a circle, and for each $n \geq 3$, let $\mathcal{P}_n$ be a regular n-gon inscribed in $\mathcal{C}$. Then $\alpha(\mathcal{C}) = \lim_{n\to\infty} \alpha(\mathcal{P}_n)$.*

Theorem 15.14 (Archimedes's Theorem). *For any circle $\mathcal{C}$ of radius r, $\alpha(\mathcal{C}) = \frac{1}{2} r \cdot \operatorname{circum}(\mathcal{C})$.*

Corollary 15.15 (Area of a Circle). *The area of a circle of radius r is πr^2.*

Constructions in Euclidean Geometry

Construction Problem 16.1 (Equilateral Triangle on a Given Segment). *Given a segment $\overline{AB}$ and a side of $\overleftrightarrow{AB}$, construct a point C on the chosen side such that $\triangle ABC$ is equilateral.*

Construction Problem 16.2 (Copying a Line Segment to a Given Endpoint). *Given a line segment $\overline{AB}$ and a point C, construct a point X such that $\overline{CX} \cong \overline{AB}$.*

Construction Problem 16.3 (Cutting Off a Segment). *Given two segments $\overline{AB}$ and $\overline{CD}$ such that $CD > AB$, construct a point E in the interior of $\overline{CD}$ such that $\overline{CE} \cong \overline{AB}$.*

Construction Problem 16.4 (Bisecting an Angle). *Given a proper angle, construct its bisector.*

Construction Problem 16.5 (Perpendicular Bisector). *Given a segment, construct its perpendicular bisector.*

Construction Problem 16.6 (Perpendicular Through a Point on a Line). *Given a line ℓ and a point $A \in \ell$, construct the line through A and perpendicular to ℓ.*

Construction Problem 16.7 (Perpendicular Through a Point Not on a Line). *Given a line ℓ and a point $A \notin \ell$, construct the line through A and perpendicular to ℓ.*

Construction Problem 16.8 (Triangle with Given Side Lengths). *Given three segments such that the length of the longest is less than the sum of the lengths of the other two, construct a triangle whose sides are congruent to the three given segments.*

Construction Problem 16.9 (Copying a Triangle to a Given Segment). *Given a triangle $\triangle ABC$, a segment $\overline{DE}$ congruent to $\overline{AB}$, and a side of $\overleftrightarrow{DE}$, construct a point F on the given side such that $\triangle DEF \cong \triangle ABC$.*

Construction Problem 16.10 (Copying an Angle to a Given Ray). *Given a proper angle $\angle ab$, a ray $\overrightarrow{c}$, and a side of $\overleftrightarrow{c}$, construct the ray $\overrightarrow{d}$ with the same endpoint as $\overrightarrow{c}$ and lying on the given side of $\overleftrightarrow{c}$ such that $\angle cd \cong \angle ab$.*

Construction Problem 16.11 (Copying a Convex Quadrilateral to a Given Segment). *Given a convex quadrilateral $ABCD$, a segment $\overline{EF}$ congruent to $\overline{AB}$, and a side of $\overleftrightarrow{EF}$, construct points G and H on the given side such that $EFGH \cong ABCD$.*

Construction Problem 16.12 (Rectangle with Given Side Lengths). *Given any two segments $\overline{AB}$ and $\overline{EF}$, and a side of $\overleftrightarrow{AB}$, construct points C and D on the given side such that $ABCD$ is a rectangle with $\overline{BC} \cong \overline{EF}$.*

Construction Problem 16.13 (Square on a Given Segment). *Given a segment $\overline{AB}$ and a side of $\overleftrightarrow{AB}$, construct points C and D on the chosen side such that $ABCD$ is a square.*

Construction Problem 16.14 (Parallel Through a Point Not on a Line). *Given a line ℓ and a point $A \notin \ell$, construct the line through A and parallel to ℓ.*

Construction Problem 16.15 (Cutting a Segment into n Equal Parts). *Given a segment $\overline{AB}$ and an integer $n \geq 2$, construct points $C_1, \dots, C_{n-1} \in \operatorname{Int}\overline{AB}$ such that $A * C_1 * \cdots * C_{n-1} * B$ and $AC_1 = C_1C_2 = \cdots = C_{n-1}B$.*

Construction Problem 16.16 (Cutting a Segment in a Rational Ratio). *Given a segment $\overline{AB}$ and a rational number x strictly between 0 and 1, construct a point $D \in \operatorname{Int}\overline{AB}$ such that $AD = x \cdot AB$.*

Construction Problem 16.17 (Geometric Mean of Two Segments). *Given two segments $\overline{AB}$ and $\overline{CD}$, construct a third segment that is their geometric mean.*

Construction Problem 16.18 (The Golden Ratio). *Given a line segment $\overline{AB}$, construct a point $E \in \operatorname{Int}\overline{AB}$ such that AB/AE is equal to the golden ratio.*

Construction Problem 16.19 (Parallelogram with the Same Area as a Triangle). *Suppose $\triangle ABC$ is a triangle and $\angle rs$ is a proper angle. Construct a parallelogram with the same area as $\triangle ABC$, and with one of its angles congruent to $\angle rs$.*

Construction Problem 16.20 (Rectangle with a Given Area and Edge). *Given a rectangle $ABCD$, a segment $\overline{EF}$, and a side of $\overleftrightarrow{EF}$, construct a new rectangle with the same area as $ABCD$, with $\overline{EF}$ as one of its edges, and with its opposite edge on the given side of $\overleftrightarrow{EF}$.*

Construction Problem 16.21 (Squaring a Rectangle). *Given a rectangle, construct a square with the same area as the rectangle.*

Construction Problem 16.22 (Squaring a Convex Polygon). *Given a convex polygon, construct a square with the same area as the polygon.*

Construction Problem 16.23 (Doubling a Square). *Given a square, construct a new square whose area is twice that of the original one.*

Construction Problem 16.24 (Center of a Circle). *Given a circle, construct its center.*

Construction Problem 16.25 (Inscribed Circle). *Given a triangle, construct its inscribed circle.*

Construction Problem 16.26 (Circumscribed Circle). *Given a triangle, construct its circumscribed circle.*

Construction Problem 16.27 (Square Inscribed in a Circle). *Given a circle and a point A on the circle, construct a square inscribed in the circle that has one vertex at A.*

Construction Problem 16.28 (Regular Pentagon Inscribed in a Circle). *Given a circle and a point A on the circle, construct a regular pentagon inscribed in the circle that has one vertex at A.*

Construction Problem 16.29 (Regular Hexagon Inscribed in a Circle). *Given a circle and a point A on the circle, construct a regular hexagon inscribed in the circle that has one vertex at A.*

Construction Problem 16.30 (Equilateral Triangle Inscribed in a Circle). *Given a circle and a point A on the circle, construct an equilateral triangle inscribed in the circle that has one vertex at A.*

Construction Problem 16.31 (Regular Octagon Inscribed in a Circle). *Given a circle and a point A on the circle, construct a regular octagon inscribed in the circle that has one vertex at A.*

Construction Problem 16.32 (Regular 15-gon Inscribed in a Circle). *Given a circle and a point A on the circle, construct a regular 15-sided polygon inscribed in the circle that has one vertex at A.*

Theorems About Euclidean Constructions

Lemma 16.33. *Every subfield of $\mathbb{R}$ contains $\mathbb{Q}$.*

Theorem 16.34. *The set K of constructible numbers is a subfield of $\mathbb{R}$.*

Theorem 16.35. *If x is a positive constructible number, then so is $\sqrt{x}$.*

Lemma 16.36. *Suppose $F \subseteq \mathbb{R}$ is a field and e is an element of F whose square root is not in F. Then a number $a + b\sqrt{e} \in F\left(\sqrt{e}\right)$ is zero if and only if $a = b = 0$.*

Lemma 16.37. *If $F \subseteq \mathbb{R}$ is a field and e is an element of F whose square root is not in F, then $F\left(\sqrt{e}\right)$ is an extension field of F containing $\sqrt{e}$.*

Theorem 16.38 (Characterization of Constructible Numbers). *A real number is constructible if and only if it is contained in some iterated quadratic extension of $\mathbb{Q}$.*

Theorem 16.39. *Let $\theta \in [0, 180]$ be arbitrary. Then there is a constructible angle with measure θ° if and only if $\cos\theta$ is a constructible number.*

Theorem 16.41. *There is no constructible 20° angle.*

Corollary 16.42. *There is no algorithm for trisecting an arbitrary angle with compass and straightedge.*

Theorem 16.43. *There is no algorithm for doubling an arbitrary cube with compass and straightedge.*

Lemma 16.44. *Every constructible number is algebraic.*

Theorem 16.45. *There is no algorithm for squaring an arbitrary circle with compass and straightedge.*

Lemma 16.46. *Let n be an integer greater than or equal to 3. Then there is a constructible regular n-gon if and only if there is a constructible angle whose measure is $360^\circ/n$.*

Theorem 16.47. *No regular 9-sided polygon can be constructed using straightedge and compass.*

Theorem 16.48 (Characterization of Constructible Regular Polygons). *A regular n-gon can be constructed with straightedge and compass if and only if n is either a power of 2 or a power of 2 times a product of distinct Fermat primes.*

Corollary 16.49. *There is no constructible regular heptagon.*

Postulates Equivalent to the Euclidean Parallel Postulate

Euclidean Postulate 1 (Euclid's Fifth Postulate). *If ℓ and ℓ' are two lines cut by a transversal t in such a way that the measures of two consecutive interior angles add up to less than $180°$, then ℓ and ℓ' intersect on the same side of t as those two angles.*

Euclidean Postulate 2 (The Equidistance Postulate). *If two lines are parallel, then each one is equidistant from the other.*

Euclidean Postulate 3 (Playfair's Postulate). *Two straight lines cannot be drawn through the same point, parallel to the same straight line, without coinciding with one another.*

Euclidean Postulate 4 (The Alternate Interior Angles Postulate). *If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.*

Euclidean Postulate 5 (Proclus's Postulate). *If ℓ and ℓ' are parallel lines, and t is a line that is distinct from ℓ but intersects ℓ, then t also intersects ℓ'.*

Euclidean Postulate 6 (The Transitivity Postulate). *If ℓ, m, and n are distinct lines such that $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.*

Euclidean Postulate 7 (Wallis's Postulate). *Given any triangle $\triangle ABC$ and any positive real number r, there exists a triangle $\triangle DEF$ similar to $\triangle ABC$, with scale factor $r = DE/AB = DF/AC = EF/BC$.*

Euclidean Postulate 8 (The Angle-Sum Postulate). *The angle sum of every triangle is equal to $180°$.*

Euclidean Postulate 9 (Clairaut's Postulate). *There exists a rectangle.*

Euclidean Postulate 10 (The Weak Angle-Sum Postulate). *There exists a triangle with zero defect.*

Neutral Geometry Theorems About Parallel Postulates

Theorem 17.1. *In the context of neutral geometry, Euclid's fifth postulate is equivalent to the Euclidean parallel postulate.*

Theorem 17.2. *In the context of neutral geometry, the equidistance postulate is equivalent to the Euclidean parallel postulate.*

Theorem 17.3. *In the context of neutral geometry, each of the four preceding postulates is equivalent to the Euclidean parallel postulate.*

Theorem 17.4. *Wallis's postulate is equivalent to the Euclidean parallel postulate.*

Lemma 17.5 (Small Angle Lemma). *In neutral geometry, suppose $\overrightarrow{FG}$ is a ray, and A is a point not on $\overleftrightarrow{FG}$. Given any positive number ε, there exists a point X in the interior of $\overrightarrow{FG}$ such that $m\angle AXF < \varepsilon$.*

Theorem 17.6. *The angle-sum postulate is equivalent to the Euclidean parallel postulate.*

Theorem 17.7 (Saccheri–Legendre). *In neutral geometry, the angle sum of every triangle is less than or equal to $180°$.*

Theorem 17.8 (Defect Addition). *In neutral geometry, suppose $\mathcal{P}$ is a convex polygon and $\overline{BC}$ is a chord of $\mathcal{P}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two convex polygons formed by $\overline{BC}$ as in the polygon splitting theorem. Then $\delta(\mathcal{P}) = \delta(\mathcal{P}_1) + \delta(\mathcal{P}_2)$.*

Corollary 17.9 (Saccheri–Legendre Theorem for Convex Polygons). *In neutral geometry, if $\mathcal{P}$ is a convex n-gon, then the angle sum of $\mathcal{P}$ satisfies $\sigma(\mathcal{P}) \le (n-2) \times 180°$. Therefore, the defect of a convex polygon is always nonnegative.*

Theorem 17.10. *Clairaut's postulate is equivalent to the Euclidean parallel postulate.*

Theorem 17.11. *The weak angle-sum postulate is equivalent to the Euclidean parallel postulate.*

Theorem 17.12 (The All-Or-Nothing Theorem). *In neutral geometry, if there exist one line ℓ_0 and one point $A_0 \notin \ell_0$ such that there are two or more distinct lines parallel to ℓ_0 through A_0, then for every line ℓ and every point $A \notin \ell$, there are two or more distinct lines parallel to ℓ through A.*

Corollary 17.13. *In every model of neutral geometry, either the Euclidean parallel postulate or the hyperbolic parallel postulate holds.*

Postulates of Hyperbolic Geometry

Postulates 1–9 of Neutral Geometry.

Postulate 10H (Hyperbolic Parallel Postulate). *For every line ℓ and every point P that does not lie on ℓ, there are at least two different lines containing P and parallel to ℓ.*

Theorems of Hyperbolic Geometry

All of the theorems of neutral geometry.

Theorem 18.1. *In hyperbolic geometry, there does not exist a rectangle.*

Theorem 18.2 (Hyperbolic Angle-Sum Theorem). *In hyperbolic geometry, every convex polygon has positive defect.*

Theorem 18.3 (AAA Congruence Theorem). *In hyperbolic geometry, if there is a correspondence between the vertices of two triangles such that all three pairs of corresponding angles are congruent, then the triangles are congruent under that correspondence.*

Theorem 18.4. *The postulates of hyperbolic geometry are not categorical.*

Theorem 18.5 (Properties of Saccheri Quadrilaterals). *Every Saccheri quadrilateral has the following properties:*

(a) *It is a convex quadrilateral.*
(b) *Its diagonals are congruent.*
(c) *Its summit angles are congruent and acute.*
(d) *Its midsegment is perpendicular to both the base and the summit.*
(e) *It is a parallelogram.*

Theorem 18.6 (Properties of Lambert Quadrilaterals). *Every Lambert quadrilateral has the following properties:*

(a) *It is a convex quadrilateral.*
(b) *Its fourth angle is acute.*
(c) *It is a parallelogram.*

Lemma 18.7 (Scalene Inequality for Quadrilaterals). *Suppose $ABCD$ is a quadrilateral in which $\angle A$ and $\angle B$ are right angles. Then $AD > BC$ if and only if $m\angle D < m\angle C$.*

Theorem 18.8. *In a Lambert quadrilateral, either side between two right angles is strictly shorter than its opposite side.*

Theorem 18.9. *In a Saccheri quadrilateral, the base is strictly shorter than the summit, and the midsegment is strictly shorter than either leg.*

Theorem 18.10. *If $\overrightarrow{AB} \mid \overrightarrow{CD}$, then $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.*

Theorem 18.11 (Existence and Uniqueness of Asymptotic Rays). *Suppose $\overrightarrow{CD}$ is a ray, and A is a point not on $\overleftrightarrow{CD}$. Then there exists a unique ray starting at A and asymptotic to $\overrightarrow{CD}$.*

Theorem 18.12 (Symmetry Property of Asymptotic Rays). *Suppose $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are two distinct rays. Then $\overrightarrow{AB} \mid \overrightarrow{CD}$ if and only if $\overrightarrow{CD} \mid \overrightarrow{AB}$.*

Theorem 18.13 (Endpoint Independence of Asymptotic Rays). *Suppose $\vec{r}$, $\vec{s}$, and $\vec{s}\,'$ are rays such that $\vec{s}\,' \subseteq \vec{s}$. Then $\vec{r} \mid \vec{s}$ if and only if $\vec{r} \mid \vec{s}\,'$.*

Theorem 18.14 (Transitive Property of Asymptotic Rays). *Suppose $\overrightarrow{AB}$, $\overrightarrow{CD}$, and $\overrightarrow{EF}$ are three rays such that $\overrightarrow{AB} \mid \overrightarrow{EF}$ and $\overrightarrow{CD} \mid \overrightarrow{EF}$, and assume that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are noncollinear. Then $\overrightarrow{AB} \mid \overrightarrow{CD}$.*

Theorem 18.15 (SA Congruence Theorem for Asymptotic Triangles). *Let $\sqsubset BACD$ and $\sqsubset B'A'C'D'$ be asymptotic triangles such that $\overline{AC} \cong \overline{A'C'}$ and $\angle C \cong \angle C'$. Then $\sqsubset BACD \cong \sqsubset B'A'C'D'$.*

Theorem 18.16 (Asymptotic Triangle Copying Theorem). *Suppose $\sqsubset BACD$ is an asymptotic triangle and $\overrightarrow{C'D'}$ is a ray. On each side of $\overleftrightarrow{C'D'}$, there is a ray $\overrightarrow{A'B'}$ asymptotic to $\overrightarrow{C'D'}$ such that $\sqsubset B'A'C'D' \cong \sqsubset BACD$.*

Theorem 18.17 (Angle-Sum Theorem for Asymptotic Triangles). *In any asymptotic triangle, the sum of the measures of the two interior angles is strictly less than $180°$.*

Theorem 18.18 (Exterior Angle Inequality for Asymptotic Triangles). *The measure of an exterior angle of an asymptotic triangle is strictly greater than the measure of its remote interior angle.*

Theorem 18.19 (AA Congruence Theorem for Asymptotic Triangles). *Let $\sqsubset BACD$ and $\sqsubset B'A'C'D'$ be asymptotic triangles. If $\angle A \cong \angle A'$ and $\angle C \cong \angle C'$, then $\sqsubset BACD \cong \sqsubset B'A'C'D'$.*

Theorem 18.20 (SA Inequality). *Suppose $\sqsubset BACD$ and $\sqsubset B'A'C'D'$ are asymptotic triangles with $\angle C \cong \angle C'$. Then $AC > A'C'$ if and only if $m\angle A < m\angle A'$.*

Theorem 18.21 (Defect Addition for Asymptotic Triangles). *Suppose $\sqsubset BACD$ is an asymptotic triangle, E is a point in $\operatorname{Int}\overline{AC}$, and $\overrightarrow{EF} \mid \overrightarrow{AB}$. Then $\delta(\sqsubset BAEF) + \delta(\sqsubset FECD) = \delta(\sqsubset BACD)$.*

Theorem 18.22 (Pasch's Theorem for Asymptotic Triangles). *Suppose $\sqsubset BACD$ is an asymptotic triangle and ℓ is a line that does not contain A or C. If ℓ intersects $\overrightarrow{AB}$, then ℓ also intersects either $\overrightarrow{CD}$ or $\overline{AC}$.*

Lemma 19.1. *Suppose ℓ and m are asymptotically parallel lines. If A is any point on ℓ and C is any point on m, then one of the rays in ℓ starting at A is asymptotic to the ray in m starting at C and lying on the same side of $\overleftrightarrow{AC}$.*

Lemma 19.2. *Suppose $\overrightarrow{AB} \mid \overrightarrow{CD}$, and E is a point on the same side of $\overleftrightarrow{AC}$ as B and D such that $\overrightarrow{AE} \neq \overrightarrow{AB}$ and $\overrightarrow{AE} \cap \overrightarrow{CD} = \varnothing$. Then $\overrightarrow{AE} * \overrightarrow{AB} * \overrightarrow{AC}$.*

Theorem 19.3 (Classification of Parallels Through a Point). *Suppose ℓ is a line and A is a point not on ℓ. Let F be the foot of the perpendicular from A to ℓ. There are exactly two lines through A that are asymptotically parallel to ℓ, and they make equal acute angle measures with $\overleftrightarrow{AF}$. A line through A is ultraparallel to ℓ if and only if it makes a larger angle measure with $\overleftrightarrow{AF}$ than the two asymptotically parallel lines, and it intersects ℓ if and only if it makes a smaller angle measure.*

Theorem 19.4 (Uniqueness of Common Perpendiculars). *If ℓ and m are parallel lines that admit a common perpendicular, then the common perpendicular is unique.*

Theorem 19.5. *Suppose ℓ and m are parallel lines. If there are two distinct points on ℓ that are equidistant from m, then ℓ and m have a common perpendicular.*

Theorem 19.6 (Ultraparallel Theorem). *Two distinct lines are ultraparallel if and only if they admit a common perpendicular.*

Theorem 19.7. *Suppose ℓ and m are two different lines. No three distinct points on ℓ are equidistant from m.*

Lemma 19.8 (Aristotle's Lemma). *Suppose $\vec{r}$ and $\vec{s}$ are rays with the same endpoint such that $\angle rs$ is acute, and c is any positive real number. Then there is a point $P \in \vec{r}$ such that $d\left(P, \overleftrightarrow{s}\right) > c$.*

Lemma 19.9. *Suppose ℓ and m are parallel lines, and $\vec{r} \subseteq \ell$ is a ray that is not asymptotic to any ray in m. For any positive real number c, there is a point $P \in \vec{r}$ such that $d(P,m) > c$.*

Theorem 19.10. *Suppose ℓ and m are ultraparallel lines, and A is the point where ℓ meets their common perpendicular.*

(*a*) *If B is any point on ℓ other than A, then $d(B,m) > d(A,m)$.*

(*b*) *If B and C are points on ℓ such that $A * B * C$, then $d(C,m) > d(B,m)$.*

(*c*) *Given any positive real number c, on each side of the common perpendicular there exists a point $P \in \ell$ such that $d(P,m) > c$.*

Theorem 19.11 (Saccheri's Repugnant Theorem). *Suppose ℓ and m are asymptotically parallel lines.*

(*a*) *If A and B are distinct points of ℓ such that $\overrightarrow{AB}$ is asymptotic to a ray in m, then $d(A,m) > d(B,m)$.*

(*b*) *If p and q are any positive real numbers, then there are points $P, Q \in \ell$ such that $d(P,m) > p$ and $d(Q,m) < q$.*