Math 583B **Research in Geometric Analysis** **Spring 2005**

LIST OF PAPERS

1. Prescribing scalar and Gaussian curvature

- J. L. Kazdan and F. W. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. **99** (1974) 14–47. This paper gives necessary and sufficient conditions on a function K on a compact 2-manifold in order that there exist a Riemannian metric whose Gaussian curvature is K.
- Thierry Aubin, *Le problème de Yamabe concernant la courbure scalaire* (French), C. R. Acad. Sci. Paris Sér. A-B **280** (1975), Aii, A721–A724. A major breakthrough in the solution of the "Yamabe problem": does every compact Riemannian manifold admit a conformal metric of constant scalar curvature? This paper solved the problem in case the manifold has dimension at least 6 and is not locally conformally flat (the so-called "local case").
- R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984) 479–495. Solved the Yamabe problem in the cases not handled by Aubin's proof (dimension less than 6 or locally conformally flat, the "global case").

2. Manifolds of positive scalar curvature

- M. Gromov and H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. **111** (1980) 423–434. The title speaks for itself.
- R. Schoen and Shing Tung Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2) **110** (1979), no. 1, 127–142. Uses a simple second variation argument to show that for a three manifold whose fundamental group contains a subgroup isomorphic to a oriented surface group of genus $g > 1$ or a noncyclic abelian group, any metric of non-negative scalar curvature is flat. This is the first appearance of the argument which later lies at the core of their minimal surface proof of the positive mass theorem.
- R. Schoen and S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), no. 1-3, 159–183. Extends the result of the paper above to dimension $n \leq 7$, and also proves that admitting positive scalar curvature is perserved under surgeries of codimension at least 3. This result underlies further work on classifying manifolds that admit metrics of positive scalar curvature.

3. Uniformization of positively-curved manifolds

- R. Schoen and S.-T. Yau, *Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature*, in "Seminar on Differential Geometry", pp. 209–228, Ann. of Math. Studies **102**, Princeton University Press, Princeton, 1982. Using minimal surface theory, shows that any complete noncompact 3-manifold with positive Ricci curvature is diffeomorphic to $\mathbb{R}^3$.
- R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982) 255–306. Uses the Ricci flow to prove that any compact 3-manifold that admits a metric with positive Ricci curvature also admits a metric with constant positive curvature, and hence is covered by the 3-sphere.

- R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geometry **24** (1986) 153–179. Extends Hamilton's theorem on three-manifolds with positive Ricci curvature to certain 4-manifolds with positive curvature, while simultaneously simplifying and clarifying the method of proof used in the 3-manifold paper.
- R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Internat. Press, Cambridge, MA, 1995. In this paper, Hamilton lays out his program for using the Ricci flow to prove the Thurston geometrization conjecture, a program eventually completed (we think) by Grisha Perelman. (Since Perelman's papers have been in circulation for more than two years and the experts are still not sure whether his proof works, those papers are not on this list, though they might someday make it here.)

4. Compactness of families of Riemannian metrics

- J. Cheeger, *Finiteness theorems for Riemannian manifolds*, Am. J. Math. **92** (1970) 61–74. Proves that in any family of Riemannian manifolds with certain bounds on curvature, volume, and diameter, there can be at most finitely many diffeomorphism types.
- M. Gromov, *Structures métriques pour les variétés riemanniennes*, Textes Mathématiques **1**, CEDIC, Paris, 1981. This is the original proof of Gromov's famous "compactness theorem": any sequence of Riemannian manifolds whose curvatures and diameters are bounded, and whose volumes are bounded below, contains a subsequence that "converges" (in a rather weak topology).
- S. Peters, *Convergence of Riemannian metrics*, Comp. Math. **62** (1987) 3–16. A more readable proof of Gromov's compactness theorem.

5. The inverse spectral problem

- B. Osgood, R. Phillips, and P. Sarnak, *Compact isospectral sets of surfaces*, J. Funct. Anal. **80** (1988) 212–234. The first partial solution to Mark Kac's famous question "Can one hear the shape of a drum?" (In other words, does the spectrum of the Laplacian on a bounded plane domain determine the domain up to isometry?) This paper proves that any family of isospectral domains in the plane (i.e. domains with the same spectrum) must be compact.
- T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. Math. **121** (1985) 169–186. Gave the first systematic construction of compact isospectral manifolds, introducing ideas that have played a central role in all subsequent isospectral examples.
- C. S. Gordon, D. L. Webb, and S. Wolpert, *One cannot hear the shape of a drum*, Bull. Amer. Math. Soc. **27** (1992) 134–138. The first counterexample to Kac's conjecture, showing that non-isometric plane domains can be isospectral.

6. Spectral asymptotics on Riemannian manifolds

- R. Seeley, *Singular integrals and boundary problems*, Amer. J. Math. **88** (1966) 781–809. The method of Hadamard is used to get information on the asymptotic behavior of eigenvalues of the Laplacian for manifolds with boundary.
- J. Chazarain, *Formule de Poisson pour les variétés riemanniennes*, Invent. Math. **1974** 65–82. Develops an asymptotic formula, generalizing the Poisson summation formula, relating

the eigenvalues of the Laplacian on a compact Riemannian manifold with the lengths of its closed geodesics.

- S. Marvizi and R. B. Melrose, *Spectral invariants of convex planar regions*, J. Differential Geom. **17** (1982) 475–502. Uses wave asymptotics to find spectral invariants of convex domains in the plane.

7. Morse theory and calculus of variations

- R. Bott, *Nondegenerate critical manifolds*, Ann. of Math. **60** (1954) 248-261. Extends Morse theory from the case of isolated critical points to the case of critical submanifolds.
- R. S. Palais and S. Smale, *A generalized Morse theory*, Bull. Amer. Math. Soc. **70** (1964) 165–172. Introduced the famous "Palais-Smale Condition C", which gives sufficient conditions for existence of critical points of functionals on infinite-dimensional spaces.
- R. S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979) 19–30. Given a function space acted on by a compact Lie group G, and a functional on the space that is G-invariant, this paper shows that, under very general conditions, functions that are critical points among G-invariant variations are actually critical among all variations. This shows that, in the presence of symmetry, calculus of variations problems can be solved by restricting attention to G-invariant objects.

8. Minimal submanifolds

- H. Federer and W. Fleming, *Normal and integral currents*, Ann. of Math. **72** (1960) 458–520. Introduces the concept of integral currents (generalized submanifolds), the main tool of geometric measure theory which is used to study minimal submanifolds.
- J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. **113** (1981) 1–24. Introduces the technique of rescaling to handle the "bubbling" phenomena that are ubiquitous in geometric nonlinear PDE's.
- R. Harvey and H. B. Lawson, *Calibrated Geometries*, Acta Math. **148** (1982) 47–157. This paper introduced an important and creative new point of view, that of "calibrations", into the subject of minimal submanifolds.

9. Geometric measure theory

- J.M. Marstrand, *he* (ϕ, s) *regular subsets of* n *space*, Trans. Amer. Math. Soc. **113** (1964), 369–392.
- D. Preiss, *Geometry of measures in* R^n*: distribution, rectifiability and densities*, Ann. of Math. **125** (1987), 537–643.
- P. Jones, *Rectifiable sets and the traveling salesman problem*, Invent. Math. **102** (1990), 1–15.

10. The Bernstein theorem for minimal surfaces

- Wendell H. Fleming, *On the oriented Plateau problem*, Rend. Circ. Mat. Palermo (2) **11** (1962) 69–90. The "right" approach to the Bernstein theorem: Any global minimal graph in R^3 must be a plane.
- James Simons, *Minimal varieties in riemannian manifolds*, Ann. Math. (2) **88** (1968) 62–105. Showed that any global minimal graph in R^8 must be a plane.

- E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969) 243–268. Constructed counterexamples of nontrivial global minimal graphs in R^9.

11. Singularities of minimal surfaces

- Leon Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983) 525–571. Proves uniqueness of tangent cones for isolated singularities. The techniques as well as the results have been very widely used.
- Luis Caffarelli, Robert Hardt, and Leon Simon, *Minimal surfaces with isolated singularities*, Manuscripta Math. 48 (1984), no. 1-3, 1–18.

12. Harmonic maps

- J. Eells, Jr., and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964) 109–160. Using heat-flow methods, they prove the existence of a harmonic map in any homotopy class from one compact Riemannian manifold to another, provided the target manifold is negatively curved.
- R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Diff. Geom. **17** (1982) 307-336. The authors prove that the Hausdorff dimension of the singular set of an energy minimizing harmonic map from an n-manifold is less than or equal to $n - 3$.
- L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Annals of Math. **118** (1983), 525-572. First steps in the development of a detailed asymptotic description of the structure of the singular set for such problems as harmonic maps or minimal surfaces.
- Fang-Hua Lin, *Gradient estimates and blow-up analysis for stationary harmonic maps*, Ann. of Math. (2) **149** (1999) 785–829. A study of the singular set of stationary harmonic maps.

13. Regularity of harmonic and p-harmonic maps

- C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta. Math. **129** (1972), 137-193. Although this paper might sound like a paper in analysis it has played a crucial role in the study of harmonic and p-harmonic maps. See, for example, the following 3 papers, which all exploit the same idea one step at a time.
- F. Helein, *Régularité des applications faiblement harmoniques entre une surface et une sphère* (French), C. R. Acad. Sci. Paris **311** (1990), 519–524. A map between manifolds is weakly harmonic if its coefficients lie in the Sobolev space H^1 are the map is a critical point for the energy functional. This paper shows that weakly harmonic maps into the sphere are actually smooth.
- F. Helein, *Regularity of weakly harmonic maps from a surface into a manifold with symmetries*, Manuscripta Math. **70** (1991), no. 2, 203–218. Generalizes the previous regularity result to maps into homogeneous Riemannian manifolds.
- F. Helein, *Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne* (French), C. R. Acad. Sci. Paris **312** (1991), 591–596. Generalizes the previous regularity results to maps into arbitrary Riemannian manifolds.

14. Yang-Mills theory and 4-manifolds

- M. Atiyah, N. Hitchin, and I. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London A **362** (1978) 425–461. This paper uses twistor methods to construct instantons (Yang-Mills connections) on S^4.
- C. Taubes, *The existence of self-dual connections on non self-dual 4-manifolds.*, J. Differential Geom. **17** (1982) 139–170. Proof of the existence of solutions to the Yang-Mills equation on a large class of 4-manifolds.
- S. K. Donaldson, *Self-dual connections and the topology of smooth 4-manifolds*, J. Differential Geom. **18** (1983) 279–315. Donaldson's spectacular result on the intersection form of simply-connected 4-manifolds with positive definite intersection form, proved by analyzing the set of solutions to the Yang-Mills equation. He showed that most topological 4-manifolds don't have smooth structures.
- S. K. Donaldson, *Polynomial invariants for smooth four-manifolds*, Topology **29** (1990), no. 3, 257–315. Extended Donaldson's results to 4-manifolds with more general intersection forms.

15. Seiberg-Witten and Gromov theories

- Edward Witten, *Monopoles and four-manifolds*, Math. Res. Lett. **1** (1994), no. 6, 769–796. Until the time of this paper, the most dramatic advances in smooth 4-manifold theory had been made using the gauge theory of the Yang-Mills equations (see "Yang-Mills theory and 4-manifolds," above). This paper revolutionized the field by introducing a new equation, the Seiberg-Witten equation, which is dramatically easier to analyze than the Yang-Mills equation.
- P. B. Kronheimer and T. S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Lett. **1** (1994), no. 6, 797–808. The first big application of Seiberg-Witten theory, to prove the Thom conjecture, which claims that holomorphic curves realize the minimum possible genus among all smooth surfaces representing a given homology class in $\mathbb{CP}^2$.
- M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347. In this paper, Gromov introduces pseudoholomorphic curves, which have turned out to be a spectacularly useful technique for studying symplectic manifolds. Among the many applications in this paper are Gromov's celebrated "nonsqueezing theorem," which says that an ϵ-ball in $\mathbb{R}^{2n}$ cannot be symplectically embedded in a δ-neighborhood of $R^{2n-2} \subset \mathbb{R}^{2n}$ for any $\delta < \epsilon$.
- Clifford Henry Taubes, *Seiberg Witten and Gromov invariants for symplectic* 4*-manifolds*, edited by Richard Wentworth, First International Press Lecture Series, 2. International Press, Somerville, MA, 2000. vi+401 pp. A collection of four papers in which Taubes proves the identity of Seiberg-Witten and Gromov invariants.

16. Conformal invariant theory

- C. Fefferman, *Parabolic invariant theory in complex analysis*, Adv. Math. **31** (1979) 131–262. Fefferman initiated the task of generalizing Weyl's invariant theory for Riemannian manifolds. In this paper, he concentrates on CR geometry, which is formally very similar to the conformal case. He described his approach to the problem, and, with a very long and difficult argument, proved that it works for some invariants in some dimensions.

- C. Fefferman and C. R. Graham, *Conformal invariants*, in "Élie Cartan et les Mathématiques d'Aujourd'hui", Astérisque, 1985, pp. 95–116. Here Fefferman's framework for constructing invariants is extended to the conformal case.
- T. N. Bailey, M. G. Eastwood, and C. R. Graham, *Invariant theory for conformal and CR geometry*, Ann. of Math. (2) **139** (1994) 491–552. The problem of describing scalar conformal invariants is completely solved in odd dimensions.

17. Complex structures and their deformations

- A. Newlander and L. Nirenberg, *Complex coordinates in almost-complex manifolds*, Ann. of Math. **65** (1957) 391–404. The proof that every integrable almost-complex manifold is actually a complex manifold.
- K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures I, II*, Ann. of Math. **67** (1958) 328–401, 403–466. The first general result on the local structure of the space of complex structures on a complex manifold.
- M. Kuranishi, *On the locally complete families of complex analytic structures*, Ann. of Math. **75** (1962) 536–577. Here Kuranishi developed a technique to completely describe the local structure of the space of inequivalent complex structures on a given complex manifold. The proof has since been simplified and extended by Kuranishi, T. Akahori, and K. Miyajima.

18. Uniformization of Kähler manifolds

- Y.-T. Siu and S.-T. Yau, *Complete Kähler manifolds with nonpositive curvature of faster than quadratic decay*, Ann. Math. **105** (1977) 225–264. A first step in a "uniformization theory" for complete Kähler manifolds in higher dimensions, showing that any negatively-curved complete Kähler manifold whose curvature decays sufficiently fast at infinity must be biholomorphic to $\mathbb{C}^n$.
- N.-M. Mok, Y.-T. Siu and S.-T. Yau, *The Poincaré-Lelong equation on complete Kähler manifolds*, Comp. Math. **44** (1981) 183–218. Shows, under the hypotheses of the Siu-Yau theorem above, that the manifold must in fact be isometric to $\mathbb{C}^n$.
- Y.-T. Siu and S.-T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980) 189–204. Proof of the Frankel conjecture: every compact Kähler manifold with positive bisectional curvature is analytically isomorphic to complex projective space. Uses harmonic map techniques.

19. Riemannian manifolds with special holonomy groups

- M. Berger, *Sur les groupes d'holonomie des variétés à connexion affine et des variétés Riemanniennes*, Bull. Soc. Math. France **83** (1955) 279–330. Shows that the only Lie groups that can occur as holonomy groups of Riemannian manifolds are $SO(n)$, $U(m)$, $SU(m)$, $Sp(k)$, $Sp(k)Sp(1)$, G_2, and $Spin(7)$.
- S. M. Salamon, *Differential geometry of quaternionic manifolds*, Ann. Scient. Ec. Norm. Sup. **19** (1986) 31–55. One of the most interesting classes of Riemannian manifolds are those with holonomy group $Sp(k)Sp(1)$, called quaternionic-Kähler manifolds. This is supposed to be a very good introduction to the subject.

- Dominic D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2 I, II*, J. Differential Geom. **43** (1996), no. 2, 291–328, 329–375. The first construction of a compact manifold with holonomy equal to the exceptional Lie group G_2.
- Dominic D. Joyce, *Compact 8-manifolds with holonomy* Spin(7), Invent. Math. **123** (1996), no. 3, 507–552. Ditto, but with holonomy Spin(7), the universal cover of SO(7).

20. Kähler-Einstein metrics

- S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Comm. Pure Appl. Math. **31** (1978) 339–411. Solves the "Calabi conjecture" for compact Kähler manifolds with non-positive first Chern class: on any such manifold, there exists a Kähler-Einstein metric in each admissible Kähler class.
- A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983) 437–443. The first known obstruction to the existence of Kähler-Einstein metrics on compact Kähler manifolds with positive first Chern class.
- S.-Y. Cheng and S.-T. Yau, *On the existence of a complete Kähler-Einstein metric on noncompact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure Appl. Math. **33** (1980) 507–544. Proves a non-compact version of Yau's theorem on the Calabi conjecture: any strictly pseudoconvex domain in $\mathbb{C}^n$ admits a complete Kähler-Einstein metric.
- Gang Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$*, Invent. Math. **89** (1987), no. 2, 225–246. This is the first of a series of papers by Tian on the difficult positive case of the Kähler-Einstein problem.

21. Geometry of strictly pseudoconvex domains in $\mathbb{C}^n$

- C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974) 1–65. Fefferman's celebrated extension theorem, which shows that biholomorphic mappings between strictly pseudoconvex domains extend smoothly to the boundary. His complicated proof has since been considerably simplified and generalized by Bell, Ligocka, Forsternič, and others.
- C. Fefferman, *Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains*, Ann. of Math. **103** (1976) 395–416. Generalizes the Poincaré metric to arbitrary strictly pseudoconvex domains in $\mathbf{C}^n$, and uses it to give a new definition of the biholomorphically invariant curves called "chains" in their boundaries.
- L. Lempert, *Solving the degenerate complex Monge-Ampère equation with one concentrated singularity*, Bull. Soc. Math. France **109** (1981) 427–454. Proved that, for any strictly convex domain in $\mathbb{C}^n$, the exponential map from an interior point is a diffemorphism onto the domain, thus providing a "canonical" parametrization for such domains. This has been applied recently by John Bland and Tom Duchamp (UW) to construct "normal forms" for convex domains and for the CR manifolds that bound them.

22. CR manifolds

- S. S. Chern and J. K. Moser, *Real Hypersurfaces in complex manifolds*, Acta Math. **133** (1974) 219–271. Solved of the "equivalence problem" for CR manifolds (abstract models of real hypersurfaces in $\mathbb{C}^n$), based on the construction of CR invariants in two different ways: by putting the power series of a real hypersurface in a "normal form" (Moser); and by constructing a canonical connection on a certain fiber bundle over the CR manifold (Chern).

- S. M. Webster, *Pseudohermitian structures on a real hypersurface*, J. Differential Geometry **13** (1978) 25–41. Carries out the analogue of the Chern connection for a CR manifold endowed with a specific choice of Levi form (a "pseudohermitian structure").
- N. Tanaka, "A Differential-Geometric Study on Strongly Pseudoconvex Manifolds", Kinokuniya Company Ltd., Tokyo, 1975. An alternative construction of the canonical connection associated with a pseudohermitian structure, carried out more or less simultaneously with and independently of Webster's.

23. Dynamical systems and symmetry

- J. E. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Reports on Math. Phys. **5** (1974) 121–130. Extends techniques used classically for dynamical systems (systems of ODE's) with abelian symmetry groups to the nonabelian case; these ideas have become useful in classical and quantum mechanics and in representation theory.
- J. M. Arms and D. C. Wilbour, *Reduction procedures for Poisson manifolds*, in "Symplectic Geometry and Mathematical Phyics", pp. 462–475, Birkhauser, Boston, 1991. Extends the notion of reduction of dynamical systems with symmetry to the case in which the orbit space has singularities.
- J. J. Duistermaat, *Global action angle coordinates*, Comm. Pure Appl. Math. **33** (1980) 687-706. Action-angle variables are particularly nice coordinates that can be used locally for dynamical systems with enough symmetry (i.e. an abelian symmetry group of the same dimension as the configuration space). This article shows that there is an obstruction to extending these coordinates globally; as an example, the obstruction is shown to be nontrivial in the case of the spherical pendulum.

24. Symplectic geometry and pseudo-holomorphic curves

- M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985) 307–347. Here Gromov introduced a new and fruitful approach to studying global invariants of symplectic manifolds: "pseudo-holomorphic curves" are maps from a Riemann surface into a symplectic manifold that are "holomorphic" with respect to some (not necessarily integrable) almost-complex structure on the symplectic manifold. As an application, he proves that no symplectic map can squash a unit ball into a "thin" slice of Euclidean space.
- T. H. Parker and J. G. Wolfson, *Pseudo-holomorphic maps and bubble trees*, J. Geom. Anal. **3** (1993) 63–98. Using Sacks & Uhlenbeck's technique of "bubbling", this paper simplifies and considerably refines Gromov's proof of compactness of sequences of pseudo-holomorphic curves, showing that the limit can be viewed as a "bubble tree", i.e. a map from a family of Riemann surfaces glued together at points into the symplectic manifold.

25. Geometric quantization

- B. Kostant, *Quantization and unitary representations, part I: prequantization*, in "Lectures in Modern Analysis and Applications, Volume III", pp. 515–538, Lecture Notes in Mathematics **170**, Springer-Verlag, New York, 1970. Introduced the basic ideas of the procedure for turning classical dynamical systems into quantum systems, now called "geometric quantization".

- B. Kostant, *Graded manifolds, graded lie theory, and prequantization*, in "Differential Geometrical methods in Mathematical Physics", Springer Lecture Notes in Math. #570, 1977, pp. 177–306. Introduced "supermanifolds," now studied by both mathematicians and physicists.

26. Existence of solutions to Einstein's equations in general relativity

- Yvonne Fourès-Bruhat, *Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires* (French), Acta Math. **88**, (1952). 141–225. In this paper, Fourès-Bruhat (who now goes by the name Choquet-Bruhat) gives the first proof of the local well-posedness of the Einstein equations.
- Demetrios Christodoulou and Sergiu Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton Mathematical Series, 41. Princeton University Press, Princeton, NJ, 1993. This is a tremendously important result on global existence of solutions to Einstein's equations. So long that it had to be published as a book, but it's really a research monograph.
- Hans Lindblad and Igor Rodnianski, *The global stability of the Minkowski space-time in harmonic gauge*, arxiv.org/math.AP/0411109. A more accessible treatment of the Christodoulou-Klainerman result.

27. Mass and black holes in general relativity

- S. W. Hawking and R. Penrose, *The singularities of gravitational collapse and cosmology*, Proc. Roy. Soc. Lond. A **300** (1970) 529–548. Proves that, under fairly general assumptions, any "physically reasonable" spacetime (4-manifold with a Lorentz metric) must have singularities (black holes).
- R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979) 45–76. Here is the original proof of the famous positive mass conjecture from physics: roughly speaking, any asymptotically flat 4-dimensional space-time with positive local mass density looks from infinity like a black hole with positive mass. Uses minimal surface theory.
- Edward Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402. A much simpler proof of the positive mass theorem, but valid only for spin manifolds.

28. The Nash-Moser inverse function theorem

- J. Nash, *The embedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956) 20–63. Proved that every Riemannian manifold can be isometrically embedded in Euclidean space, and introduced the iterative technique that was later formalized as the Nash-Moser implicit function theorem.
- J. Moser, *A rapidly convergent iteration method and nonlinear PDE's I, II*, Ann. Scuola Norm. Sup. Pisa **20** (1966) 265–315, 499–535. Moser's version of the Nash-Moser inverse function theorem, now called Moser iteration.

- R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. **7** (1982) 65–222. A very general treatment of the celebrated Nash-Moser theorem, which extends the inverse function theorem to spaces of C^∞ functions.

29. Elliptic PDE's and pseudodifferential operators

- L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Appl. Math. **6** (1953) 337–394. This is essentially Nirenberg's thesis, in which he solved the Weyl problem (given a positively-curved Riemannian metric on the 2-sphere, find an isometric embedding into $\mathbf{R}^3$) and the closely-related Minkowski problem. He derived some fundamental results on variable-coefficient elliptic PDE's, introducing techniques that were later formalized into the theory of pseudo-differential operators.
- J. J. Kohn and L. Nirenberg, *An algebra of pseudo-differential operators*, Invent. Math. **24** (1974) 65–82. Here the concept of pseudo-differential operators was formally introduced. These operators are now the main tool in studying elliptic and subelliptic PDE's.
- R. B. Melrose, *Transformation of Boundary Problems*, Acta Math. **147** (1981) 149–236. Introduces a new pseudo-differential operator calculus, that of "totally characteristic" operators, to handle boundary problems for differential operators that degenerate along the boundary.

30. Fourier integral operators

- P. D. Lax, *Asymptotic solutions of oscillatory initial value problems*, Duke Math. J. **24** (1957) 627–646. Here the method of geometric optics is formalized and used to solve hyperbolic PDE's.
- L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968) 193–218. Geometric optics is generalized and developed into a precursor to Fourier integral operators, with applications to spectral theory.
- L. Hörmander, *Fourier integral operators I*, Acta Math. **127** (1971) 79–183. This paper and the next formally introduced the concept of Fourier integral operators, now widely used to study hyperbolic PDE's.
- J. J. Duistermaat and L. Hörmander, *Fourier integral operators II*, Acta Math. **128** (1972) 183–269.

31. Local solvability of PDE's

- H. Lewy, *An example of a smooth linear partial differential equation without solution*, Ann. of Math. **66** (1957) 155–158. A simple first-order linear partial differential equation that has no solution in any open set. This started the whole subject of local solvability.
- L. Nirenberg and F. Trèves, *On local solvability of partial differential equations, Part I: necessary conditions*, Comm. Pure Appl. Math. **23** (1970) 1–38; *Part II: sufficient conditions*, Comm. Pure Appl. Math. **23** (1970) 459–510. Characterized locally solvable partial differential operators.

32. The $\overline{\partial}$ problem

- J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds, I*, Ann. Math. **78** (1963) 112–148; *II*, **79** (1964) 450–472. Solved the $\overline{\partial}$-Neumann problem: on a strictly pseudoconvex

domain $\Omega \subset \mathbb{C}^n$, given a smooth $\overline{\partial}$-closed differential form η on $\overline{\Omega}$, find a smooth form ω on $\overline{\Omega}$ such that $\overline{\partial}\omega = \eta$. This has many applications to several complex variables, including solving the Levi problem of characterizing domains of holomorphy, and solving the Cousin problems of extending holomorphic functions.

- J. J. Kohn and L. Nirenberg, *Non-coercive boundary problems*, Comm. Pure Appl. Math. **18** (1965) 443–492. Extends Kohn's work on the $\overline{\partial}$-Neumann problem to a very general class of elliptic boundary problems that are not uniformly elliptic up to the boundary.
- L. Hörmander, *L^2 estimates and existence theorems for the $\overline{\partial}$ operator*, Acta Math. **113** (1965) 89–152. Presents an alternative solution to the $\overline{\partial}$ problem.

33. Subelliptic PDE's in CR geometry

- J. J. Kohn, *Boundaries of complex manifolds*, in "Proc. Conf. on Complex Analysis", Minneapolis 1964 Springer-Verlag, New York, 1965, pp. 81–94. Proves a "Hodge theorem" for the boundary Cauchy-Riemann operator $\overline{\partial}_b$ on a CR manifold, analogous to Kohn's earlier results for $\overline{\partial}$ on strictly pseudoconvex domains. One of the first systematic treatments of "subelliptic" differential operators.
- L. Hörmander, *Hypoelliptic second-order differential equations*, Acta Math. **119** (1967) 147–171. Shows that a real second-order partial differential operator that can be written as a sum of squares of vector fields is hypoelliptic provided the vector fields and their brackets span the tangent space at each point. This is the prototype of subelliptic behavior which is now extremely important in several complex variables.
- G. B. Folland and E. M. Stein, *Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **28** (1974) 429–522. Greatly extends and generalizes the results of Kohn on subellipticity of the $\overline{\partial}_{\mathrm{b}}$ complex.

34. Index theorems for elliptic operators on manifolds

- M. Atiyah and I. M. Singer, *The index of elliptic operators I*, Ann. Math. **87** (1968) 484–530. This paper and the next four gave the initial proof of the celebrated Atiyah-Singer index theorem, which relates the index of an elliptic operator on a manifold (i.e. the dimension of the kernel minus the codimension of the image) to topological invariants.
- M. Atiyah and G. B. Segal, *The index of elliptic operators II*, Ann. Math. **87** (1968) 531–545.
- M. Atiyah and I. M. Singer, *The index of elliptic operators III*, Ann. Math. **87** (1968) 546–604; *IV*, **93** (1971) 119–138; *V*, **93** (1971) 139–149.
- M. Atiyah, R. Bott, and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973) 279–330; *Errata*, **28** (1975) 277–280. Uses classical invariant theory to give a greatly simplified proof of a key step in the Atiyah-Singer index theorem. Good overview of the history of the index theorem and its proof.

35. Rigidity of Riemannian Manifolds

- K. Corlette, *Archimedean superrigidity and hyperbolic geometry*, Ann. of Math. (2) **135** (1992), no. 1, 165–182. The first paper in which harmonic maps are used to study questions of rigidity. It inspired the Gromov-Schoen paper below, which made the subject popular in the early 90s.

- M. Gromov and R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Inst. Hautes Études Sci. Publ. Math. No. 76 (1992), 165-246.
- M. Gromov, *Filling Riemannian manifolds*, J. Differential Geom. **18** (1983) 1–147. In this seminal paper, Gromov proves (among many other things) that any compact subdomain of Euclidean space is boundary rigid by showing that the "filling volume" of the boundary is greater than or equal to the volume of the domain. The rigidity follows by showing that equality implies flatness. Like most of Gromov's papers this is hard to read.
- C. Croke, *Rigidity and the distance between boundary points*, J. Differential Geom. **33** (1991) 445–464. A simplification of Gromov's proof of boundary rigidity.
- Dmitri Burago and Sergei Ivanov, *Riemannian tori without conjugate points are flat*, Geom. Funct. Anal., **4** (1994) 259–269. A proof of Hopf's conjecture that any metric without conjugate points on the n-torus must be flat.