Math 120 - Winter 2016
Final Exam
March 12, 2016

Name: ______________________ Student ID no. : ______________

Signature: ______________________________ Section: __________

1	13	
2	13	
3	13	
4	13	
5	12	
6	12	
7	12	
8	12	
Total	100	

- This exam consists of EIGHT problems on NINE pages, including this cover sheet.
- Show all work for full credit.
- You may use a scientific calculator during this exam. Graphing calculators are not permitted. Other electronic devices are not allowed, and should be turned off and put away for the duration of the exam.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- If you write on the back of the page, please indicate that you have done so!
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 170 minutes to complete the exam.

1. For parts (a) through (c), consider the function $f(x) = 2 \cdot 3^{6-2x}$.

   (a) **[5 points]** Write a formula for the inverse function, $f^{-1}(x)$.

$$x = 2 \cdot 3^{6-2y}$$

$$\frac{x}{2} = 3^{6-2y}$$

$$\log_3\left(\frac{x}{2}\right) = 6-2y$$

$$2y = 6 - \log_3\left(\frac{x}{2}\right)$$

$$\boxed{f^{-1}(x) = 3 - \frac{1}{2}\log_3\left(\frac{x}{2}\right)}$$

   (b) **[4 points]** Suppose $f(a^2+1) = 54$. Find all possible values of a.

$$f(a^2+1) = 54$$

$$a^2+1 = f^{-1}(54) = 3 - \frac{1}{2}\log_3\left(\frac{54}{2}\right)$$

$$a^2+1 = \frac{3}{2}$$

$$\boxed{a = \pm\sqrt{\frac{1}{2}}}$$

   (c) **[4 points]** Write $f(x)$ in standard exponential form.

   (That is, write it as $f(x) = A_0 b^x$ for some constants A_0 and b.)

$$f(x) = 2 \cdot 3^{6-2x}$$

$$= 2 \cdot 3^6 \cdot 3^{-2x}$$

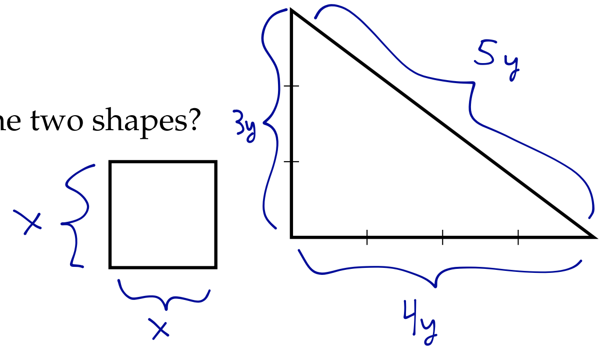
$$= (1458)(3^{-2})^x$$

$$\boxed{= (1458)\left(\frac{1}{9}\right)^x}$$

2. **[13 points]** You have 8 feet of wire which you would like to use to make two shapes: a *square*, and a *right triangle*, where one leg of the right triangle is three-fourths the length of the other leg. (You have to use all the wire.)

For example, they might look like this:

What is the **minimum possible total area** of the two shapes?



$$\text{Area} = x^2 + \tfrac{1}{2}(3y)(4y)$$
$$= x^2 + 6y^2$$

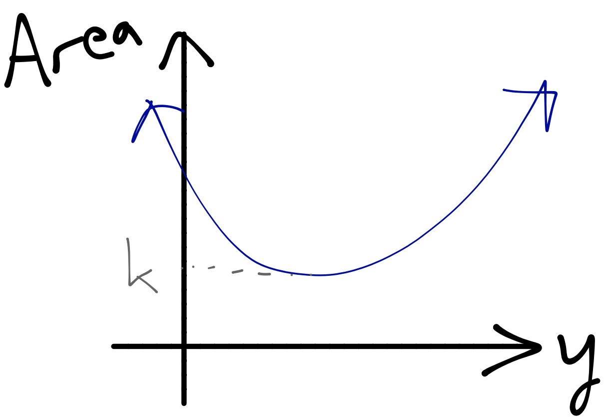
Constraint: $8 = 4x + 12y$

$$\downarrow$$

$$x = 2 - 3y$$

$$\text{Area} = (2-3y)^2 + 6y^2$$

$$\text{Area} = 15y^2 - 12y + 4$$



$$k = 4 - \frac{(-12)^2}{4(15)} = 4 - \frac{144}{60}$$

$$= \boxed{1.6 \text{ sq. ft}}$$

3. **[13 points]** Sinead decides to track the noise level where she's standing at Sasquatch, and finds that the volume (measured in decibels) is a sinusoidal function of time.

Right now, the volume is at its maximum: 80 decibels.

The volume will decrease for the next 14 minutes until it hits its minimum: 44 decibels.

Over the next hour (starting now), for how long will the volume be above 70 decibels?

$f(t)$ = volume (in dB) t minutes from now.

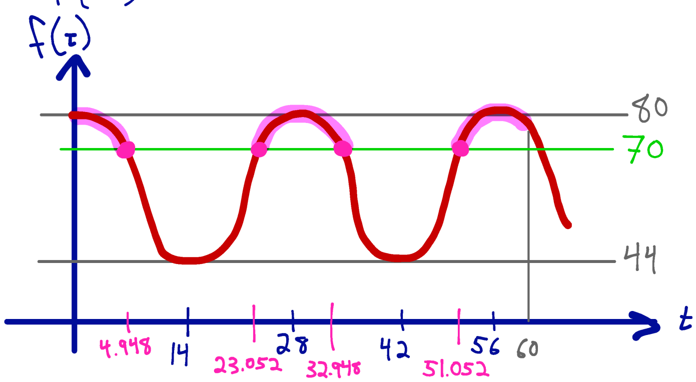
$$f(t) = A\sin\left(\frac{2\pi}{B}(t-C)\right)+D.$$

$$A = \frac{80-44}{2} = 18 \qquad C = 0 - \frac{28}{4} = -7 \text{ (or 21, or...)}$$

$$B = 28 \qquad D = \frac{80+44}{2} = 62$$

(time from max to min)*2

$$f(t) = 18\sin\left(\frac{2\pi}{28}(t+7)\right)+62$$



List of solutions:

principal: −4.948 (+28)

symmetry: 4.948 (+28)

23.052 (+28)

32.948 (+28)

51.052 (+28)

60.948

⋮

Solutions?

$$18\sin\left(\frac{2\pi}{28}(t+7)\right)+62 = 70$$

$$\sin\left(\frac{2\pi}{28}(t+7)\right) = \frac{4}{9}$$

$$\frac{2\pi}{28}(t+7) = \sin^{-1}\left(\frac{4}{9}\right)$$

$$t = -7 + \frac{28}{2\pi}\sin^{-1}\left(\frac{4}{9}\right)$$

$$t = -4.948 \leftarrow \text{principal solution}$$

symmetry solution $= 2C + \frac{B}{2} - p = 4.948$

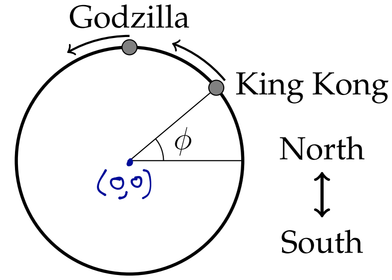
Time above 70 dB:

$$(4.948-0)$$
$$+(32.948-23.052)$$
$$+(60-51.052)$$
$$= \boxed{23.792 \text{ min}}$$

4. Godzilla and King Kong are running around the Earth, up the prime meridian and down its antimeridian, as shown below. The radius of Earth is 4000 miles.

   (a) **[3 points]** Godzilla begins at the north pole. It takes him 5 hours to run one full circle around the world.

   What is Godzilla's speed, in miles per hour?



$V = \omega r$

$$\boxed{= \left(\frac{2\pi}{5}\right)(4000) \approx 5026.5 \text{ miles per hour}}$$

   (b) **[5 points]** King Kong starts at an angle ϕ north of the equator and runs in the same direction as Godzilla at a speed of 6000 miles per hour.

   He passes Godzilla after 3 hours. What's ϕ? Give your answer in radians.

$$\phi + 3\omega_K = \frac{\pi}{2} + 3\omega_G \qquad \omega_K = \frac{V}{r} = \frac{6000}{4000} = 1.5 \text{ rad/hour}$$

$$\phi = \frac{\pi}{2} + 3\left(\frac{2\pi}{5}\right) - 3(1.5)$$

$$= \boxed{0.841 \text{ rad.}}$$

   (c) **[5 points]** After 14 hours, who is farther north: Godzilla or King Kong?

$$y_G = 4000 \sin\left(\frac{\pi}{2} + \left(\frac{2\pi}{5}\right)(14)\right) = 1236.1$$

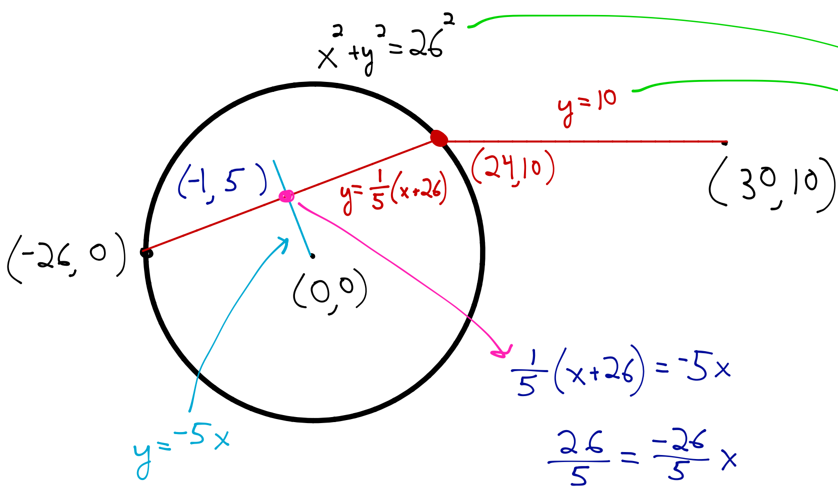
$$y_K = 4000 \sin(0.841 + (1.5)(14)) = 599.5$$

$\boxed{\text{Godzilla is farther north.}}$

5. **[12 points]** Nessie stands 30 miles east and 10 miles north of the center of a circular lake with radius 26 miles.

She walks due west in a straight line until she hits the edge of the lake. Then, she swims in a straight line towards the westernmost point of the lake.

When she is closest to the center of the lake, how far is she from her starting position?



$x^2+10^2=26^2$

$x^2=576$

$x=24$

$\frac{1}{5}(x+26)=-5x$

$\frac{26}{5}=\frac{-26}{5}x$

$x=-1$

Closest pt: $(-1,5)$

Distance from $(-1,5)$ to $(30,10)$:

$$\sqrt{31^2+5^2}=\sqrt{986}\approx \boxed{31.401 \text{ miles}}$$

6. **[4 points per part]**

(a) A goblin starts at $(-4, 2)$ at time $t = 0$ and walks in a straight line at a constant speed towards the point $(5, -3)$, reaching it in 5 seconds.

Find parametric equations for the goblin's coordinates after t seconds.

$x_0 = -4 \quad y_0 = 2$

$x_1 = 5 \quad y_1 = -3$

$\Delta x = 9 \quad \Delta y = -5$

$\Delta t = 5$

$$\boxed{x = -4 + \frac{9}{5}t \qquad y = 2 + \frac{-5}{5}t}$$

(b) Meanwhile, a ghoul starts at $(7, 3.2)$ at time $t = 0$ and walks in a straight line towards the point $(-5, 1)$ at a constant speed of 4 units per second.

Find parametric equations for the ghoul's coordinates after t seconds.

$x_0 = 7 \quad y_0 = 3.2$

$x_1 = -5 \quad y_1 = 1$

$\Delta x = -12 \quad \Delta y = -2.2$

$\Delta t = 3.05$

$\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = 12.2$

$\Delta t = \frac{12.2}{4} = 3.05$ seconds

$$\boxed{x = 7 + \frac{-12}{3.05}t \qquad y = 3.2 + \frac{-2.2}{3.05}t}$$

(c) A zombie stands at the origin. After 1.5 seconds, who is closer to the zombie: the goblin or the ghoul?

Goblin at:

$x = -4 + \frac{9}{5}(1.5) = -1.3$

$y = 2 - 2 = 0.5$

Distance from origin: $\sqrt{(-1.3)^2 + (0.5)^2} \approx 1.39$

Ghoul at:

$x = 7 + \frac{-12}{3.05}(1.5) = 1.10$

$y = 3.2 - \frac{2.2}{3.05}(1.5) = 2.12$

Distance from origin $= \sqrt{1.10^2 + 2.12^2} \approx 2.39$

$\boxed{\text{Goblin is closer.}}$

7. **[12 points]** Frank is engaged in an election campaign. The percentage of the vote he earns in the New Hampshire primary is a *linear-to-linear rational function* of the number of advertising dollars he spends.

If he spends 1 million dollars, he'll earn 34% of the vote.

If he spends 4 million dollars, he'll earn 49% of the vote.

As his advertising spending increases, his vote share will approach (but not reach) 65%.

How much does he need to spend to earn 55% of the vote?

$$f(x) = \frac{ax+b}{x+d} = \text{percent of vote after spending \$x mill.}$$

$$\boxed{a = 65}$$

$$\frac{a+b}{1+d} = 34 \rightarrow 65 + b = 34 + 34d$$

$$\frac{4a+b}{4+d} = 49 \rightarrow -(260 + b = 196 + 49d)$$

$$-195 = -162 - 15d$$

$$15d = 33$$

$$\boxed{d = 2.2}$$

$$65 + b = 34 + 34(2.2)$$

$$\boxed{b = 43.8}$$

$$f(x) = \frac{65x + 43.8}{x + 2.2} = 55$$

$$65x + 43.8 = 55x + 121$$

$$10x = 77.2$$

$$\boxed{x = \$7.72 \text{ million}}$$

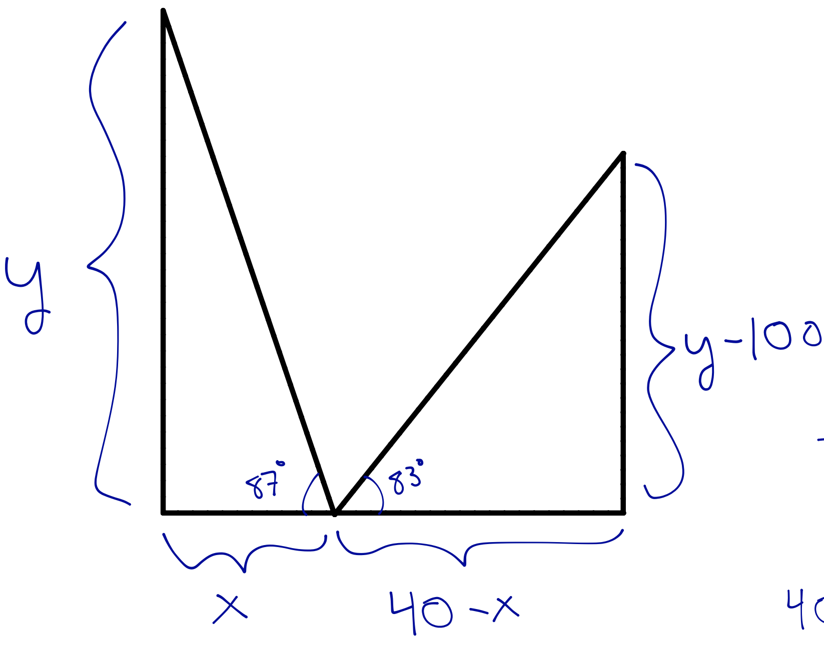
8. **[12 points]** Two buildings, creatively named Tower A and Tower B, stand across from each other along a street. Tower A is 100 feet taller than Tower B.

The buildings are 40 feet apart. You are standing somewhere in the street between them (but not necessarily halfway), carefully dodging traffic while you solve a math problem.

From where you stand, the top of Tower A is at an angle of $87°$ above the horizontal, and the top Tower B is at an angle of $83°$ above the horizontal.

How tall is Tower A?

(Ignore your own height in this problem.)



$$\tan(87°) = \frac{y}{x} \rightarrow y = x\tan(87°)$$

$$\tan(83°) = \frac{y-100}{40-x}$$

$$\downarrow$$

$$\tan(83°)(40-x) = x\tan(87°) - 100$$

$$\downarrow$$

$$40\tan(83°) + 100 = x(\tan(87°) + \tan(83°))$$

$$x = \frac{40\tan(83°) + 100}{\tan(87°) + \tan(83°)}$$

$$y = \frac{40\tan(83°) + 100}{\tan(87°) + \tan(83°)}\tan(87°) = \boxed{298.4 \text{ ft}}$$