1. (10 total points) Evaluate the following definite integrals. Simplify and box your answers.

   (a) (5 points) $\int_0^{\pi/4} \tan^2\theta \sec^4\theta \, d\theta$

$$= \int_0^{\pi/4} \tan^2\theta \sec^2\theta \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \tan^2\theta (\tan^2\theta + 1) \sec^2\theta \, d\theta$$

$u = \tan\theta$
$du = \sec^2\theta \, d\theta$

$$= \int_0^1 u^2(u^2+1)\, du = \int_0^1 u^4 + u^2 \, du$$

$$= \frac{1}{5}u^5 + \frac{1}{3}u^3 \Big|_0^1 = \frac{1}{5} + \frac{1}{3} = \boxed{\frac{8}{15}}$$

   (b) (5 points) $\int_0^1 x^2 \arcsin(x)\, dx$

IBP: $u = \arcsin x \quad dv = x^2 dx$
$du = \frac{1}{\sqrt{1-x^2}} dx \quad v = \frac{1}{3}x^3$

$$= \frac{1}{3}x^3 \arcsin x \Big|_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{\sqrt{1-x^2}} dx$$

u-sub: $u = 1 - x^2$
$du = -2x\,dx$
(or trig sub $u = \sin\theta$)

$$= \left(\frac{1}{3}(1)\left(\frac{\pi}{2}\right) - 0\right) - \frac{1}{3}\int_1^0 \frac{(1-u)}{\sqrt{u}}\left(-\frac{1}{2}\right) du$$

$$= \frac{\pi}{6} - \frac{1}{6}\int_0^1 u^{-1/2} - u^{1/2}\, du$$

$$= \frac{\pi}{6} - \frac{1}{6}\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right)\Big|_0^1 = \frac{\pi}{6} - \frac{1}{6}\left(2 - \frac{2}{3}\right)$$

$$= \boxed{\frac{\pi}{6} - \frac{2}{9}}$$

2. (10 points) Evaluate the following indefinite integrals.

   (a) (5 points) $\int \sqrt{10x - x^2}\, dx$

complete the square

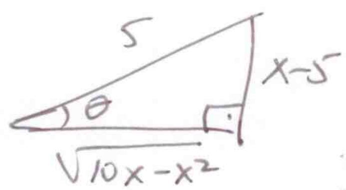
$$= \int \sqrt{25-(x-5)^2}\, dx$$

trig sub: $x-5 = 5\sin\theta$
$dx = 5\cos\theta\, d\theta$

$$= \int \sqrt{25\cos^2\theta}\; 5\cos\theta\, d\theta$$

$$= 25\int \cos^2\theta\, d\theta = \frac{25}{2}\int 1+\cos(2\theta)\, d\theta$$

$$= \frac{25}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C = \frac{25}{2}\left(\theta + \sin\theta\cos\theta\right) + C$$



$$= \frac{25}{2}\left[\arcsin\left(\frac{x-5}{5}\right) + \frac{(x-5)\sqrt{10x-x^2}}{25}\right] + C$$

$$= \boxed{\frac{25}{2}\arcsin\left(\frac{x-5}{5}\right) + \frac{1}{2}(x-5)\sqrt{10x-x^2} + C}$$

$\sin\theta = \frac{x-5}{5}$

$\cos\theta = \frac{\sqrt{10x-x^2}}{5}$

   (b) (5 points) $\int \frac{x-1}{x^3+x}\, dx$

$$\frac{x-1}{x^3+x} = \frac{x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x-1 = (A+B)x^2 + Cx + A \implies A=-1,\ C=1,\ B=-A=1$$

$$\int \frac{x-1}{x^3+x}\, dx = \int \frac{-1}{x} + \frac{x+1}{x^2+1}\, dx$$

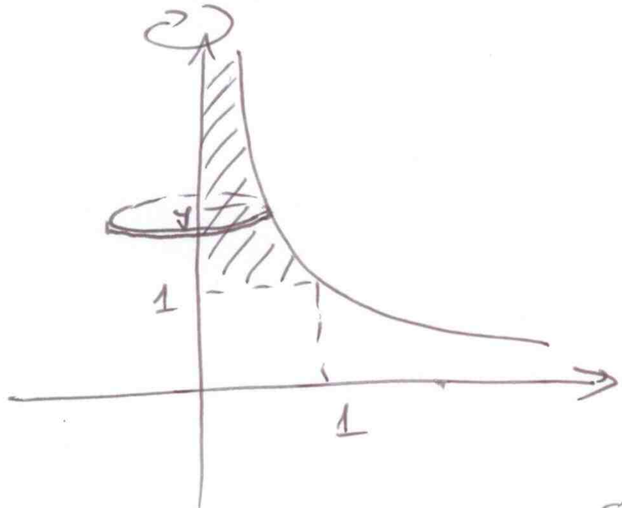
$$= \int \frac{-1}{x} + \frac{x}{x^2+1} + \frac{1}{x^2+1}\, dx$$

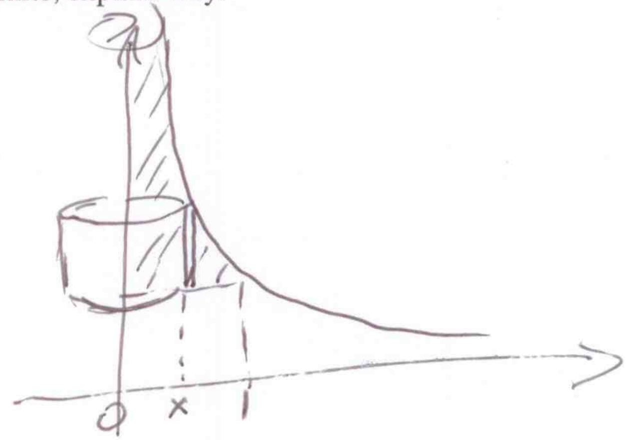
$$= \boxed{-\ln|x| + \frac{1}{2}\ln(x^2+1) + \arctan(x) + C}$$

3. (10 points) Consider the infinite region in the first quadrant of the xy-plane, above the line $y = 1$, and to the left of the curve

$$y = \frac{1}{\sqrt{x}}$$

Rotate this region about the y-axis, and determine whether the volume of the resulting solid is finite or infinite. If it is finite, find the volume. If it is infinite, explain why.





Using Disks:

$$V = \int_1^\infty \pi \left(\frac{1}{y^2}\right)^2 dy$$

$$= \lim_{t \to \infty} \int_1^t \pi y^{-4} dy$$

$$= \pi \lim_{t \to \infty} \left(-\frac{1}{3y^3}\right)\Big|_1^t$$

$$= \pi \lim_{t \to \infty} \left(-\frac{1}{3t^3} + \frac{1}{3}\right)$$

$$= \pi \left(0 + \frac{1}{3}\right)$$

$$= \boxed{\frac{\pi}{3}}$$

OR Using Shells:

$$V = \int_0^1 2\pi\, x \left(\frac{1}{\sqrt{x}} - 1\right) dx$$

$$= 2\pi \int_0^1 \sqrt{x} - x \, dx$$

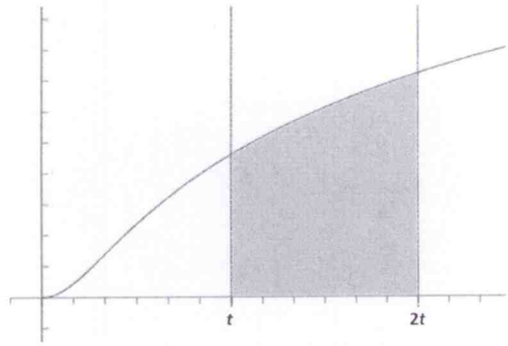
$$= 2\pi \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^2\right)\Big|_0^1$$

$$= 2\pi \left(\frac{2}{3} - \frac{1}{2}\right)$$

$$= 2\pi \frac{1}{6}$$

$$= \boxed{\frac{\pi}{3}}$$

4. (10 total points) The figure on the right shows a region bounded to the left by the line $x = t$, to the right by $x = 2t$, on the top by the curve $y = \ln(x^2+1)$, and on the bottom by the x-axis.



(a) (2 points) Set up an integral for the area $A(t)$ of this region. DO NOT EVALUATE the integral.

$$A_{(t)} = \int_t^{2t} \ln(x^2+1)\,dx$$

(b) (4 points) Compute $A'(1)$

$$A_{(t)} = -\int_0^t \ln(x^2+1)\,dx + \int_0^{2t} \ln(x^2+1)\,dx$$

Applying FTC I (& Chain Rule):

$$A'(t) = -\ln(t^2+1) + \ln(4t^2+1)\cdot 2$$

Evaluating at $t=1$

$$A'(1) = \boxed{-\ln(2) + 2\ln 5} = \boxed{\ln\left(\frac{25}{2}\right)}$$

(c) (4 points) Set up an integral for the arc length $L(t)$ of the top boundary of this region (that is, the arc length of the curve $y = \ln(x^2+1)$, $t \le x \le 2t$). DO NOT EVALUATE the integral.

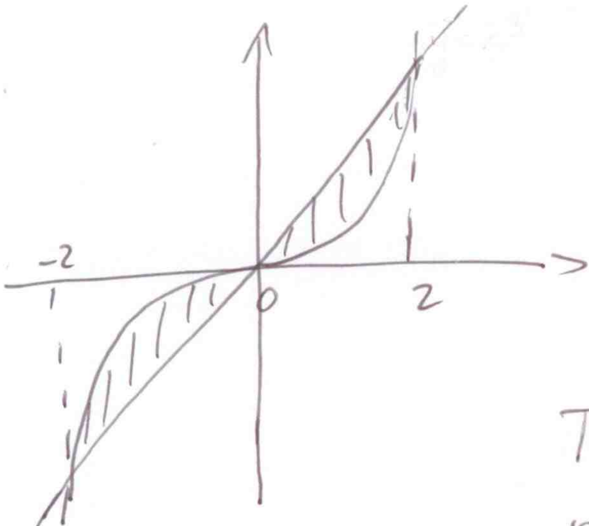
$$\frac{dy}{dx} = \frac{2x}{x^2+1}$$

$$\boxed{L = \int_t^{2t} \sqrt{1 + \left(\frac{2x}{x^2+1}\right)^2}\,dx}$$

5. (10 points) The curves:

$$y = x^3 \qquad \text{and} \qquad y = 4x$$

enclose two regions in the plane. Find the total area of these regions.



Intersection points:

$x^3 = 4x$

$x(x^2-4) = 0$

$x = 0,\ x = \pm 2$

$$\text{Total area} = \int_{-2}^{2} |4x - x^3|\,dx$$

$$= \int_{-2}^{0} (x^3 - 4x)\,dx + \int_{0}^{2} (4x - x^3)\,dx$$

$$= 2 \int_{0}^{2} (4x - x^3)\,dx$$

$$= 2\left[2x^2 - \tfrac{1}{4}x^4\right]\Big|_0^2$$

$$= 2\,[8-4] = \boxed{8}$$

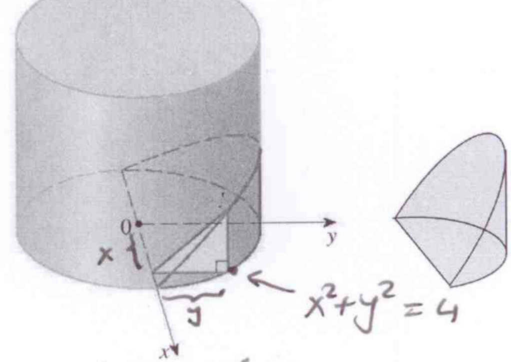
6. (10 points) A wedge is cut out of a right cylinder of radius 2 by two planes. One plane is horizontal, perpendicular to the axis of the cylinder. The other plane intersects the first at an angle of 30° along a diameter of the cylinder.

Compute the volume of the wedge.

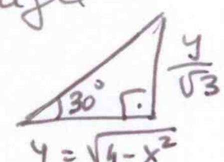
This is <u>not</u> a solid of revolution!

Use the slicing method:

$$V = \int_a^b A(x)\,dx$$



If we slice as indicated in the picture, each slice is a right triangle w/ base y & angle 30° so $\tan 30° = \frac{h}{y}$



i.e. height $h = y \tan 30° = y/\sqrt{3}$

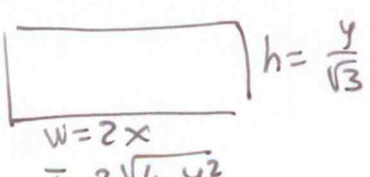
$$\therefore h = (\sqrt{4-x^2})/(\sqrt{3})$$

$$A(x) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(\sqrt{4-x^2})\left(\frac{1}{\sqrt{3}}\sqrt{4-x^2}\right) = \frac{1}{2\sqrt{3}}(4-x^2)$$

$$V = \int_{-2}^{2} \frac{1}{2\sqrt{3}}(4-x^2)\,dx = \cancel{2}\int_0^2 \frac{1}{\cancel{2}\sqrt{3}}(4-x^2)\,dx$$

$$= \frac{1}{\sqrt{3}}\left(4x - \frac{x^3}{3}\right)\Big|_0^2 = \frac{1}{\sqrt{3}}\left(8 - \frac{8}{3}\right) = \boxed{\frac{16}{3\sqrt{3}}} = \boxed{\frac{16\sqrt{3}}{9}}$$

Alternatively, we could slice $\perp$ on the y-axis get rectangular slices:



$$\Rightarrow V = \int_0^2 2\sqrt{4-y^2}\,\frac{y}{\sqrt{3}}\,dy$$

$$= [\ldots]$$

7. (10 total points) A calculus student lifts a bag of sand, beginning at time $t = 0$, at a constant rate of 0.20 meters per second, from the ground to a height of 2 meters above ground. The initial mass of the sand in the bag is 10 kilograms (the mass of the bag is negligible) but sand drains from a hole in the bottom of the bag at the variable rate of $\frac{1}{1+t}$ kilograms per second. Recall that the gravitational acceleration is $g = 9.8\ m/s^2$.

   (a) (4 points) Find the mass $m(t)$ of sand in the bag at time t seconds.

We know: $m(0) = 10$ kg & $m'(t) = \frac{-1}{1+t}$

Integrating: $m(t) = -\ln(1+t) + C$

$m(0) = 0 + C = 10 \Rightarrow C = 10$

$\therefore \boxed{m(t) = 10 - \ln(1+t)}$

   (b) (6 points) Set up an integral for the work done in lifting the sand. DO NOT EVALUATE the integral.

$W = \int_0^2 F(y)\,dy$

Converting time t to height y: $y = 0.2t \Rightarrow t = 5y$

so mass $m(y) = 10 - \ln(1+5y)$

The weight is $F(y) = 9.8\,m(y) = 9.8(10 - \ln(1+5y))$

$$\boxed{W = \int_0^2 9.8(10 - \ln(1+5y))\,dy}$$

OR, in terms of t: $t=0$ to $t = \frac{2m}{0.2m/s} = 10$ sec.

& $t = 5y \Rightarrow dt = 5\,dy \Rightarrow dy = \frac{1}{5}dt$

$$\boxed{W = \int_0^{10} 9.8(10 - \ln(1+t))\frac{1}{5}\,dt}$$

8. (10 points) Find the solution of the differential equation below that satisfies the given initial condition. Write your solution y as an explicit function of x, that is, your answer should be expressed in the form $y = f(x)$.

$$y' = xe^x + xy^2e^x \qquad y(0) = 0$$

$$\frac{dy}{dx} = xe^x + xy^2e^x = xe^x(1+y^2)$$

Separating the variables and integrating:

$$\int \frac{1}{1+y^2}\,dy = \int xe^x\,dx$$

IBP $u = x \quad dv = e^x dx$
$du = dx \quad v = e^x$

$$\arctan(y) = xe^x - \int e^x dx$$
$$= xe^x - e^x + C$$

Using the initial value to "fix" C:

$$\arctan(0) = 0 - 1 + C \Rightarrow C = 1$$

$$\therefore \arctan(y) = xe^x - e^x + 1$$

$$\boxed{y = \tan(xe^x - e^x + 1)}$$

9. (10 points) A tank contains 100 liters of fresh water. Water containing s grams of salt per liter enters the tank at the rate of 10 liters/minute, and the well-mixed solution leaves at the same rate.

   (a) (3 points) Let $y(t)$ denote the amount of salt in the tank at time t minutes. Write down a differential equation for $y(t)$. (This equation will contain the constant s.)

IN: s gr/L at 10 L/min $\Rightarrow$ rate in $= 10s$ gr/min

OUT: $\frac{y}{100}$ at 10 L/min $\Rightarrow$ rate out $= \frac{y}{10}$ gr/min.

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

$$\boxed{\frac{dy}{dt} = 10s - \frac{y}{10}}$$

   (b) (7 points) Suppose that after 20 minutes the concentration of salt in the tank is 3 grams/liter. Compute s. Leave your answer in exact form.

$$\frac{dy}{dt} = \frac{100s - y}{10}$$

$$\int \frac{1}{100s-y}\,dy = \int \frac{1}{10}\,dt$$

$$-\ln|100s-y| = \int \frac{1}{10}\,dt$$

$$\ln|100s-y| = -\frac{1}{10}t + C$$

$$|100s-y| = e^{-0.1t}\cdot e^{C} = Ae^{-0.1t}$$

$$y - 100s = \pm A e^{-0.1t}$$

$$y = 100s \pm A e^{-0.1t} = 100s + Be^{-0.1t}$$

At $t=0, y=0 \Rightarrow 100s + B = 0 \Rightarrow B = -100s$

$$y = 100s\left(1 - e^{-0.1t}\right)$$

At $t=20$, $y = 300$ g :

(conc. $= 3 = \frac{y}{100} \Rightarrow y = 300$)

$$300 = 100s\left(1 - e^{-0.1(20)}\right)$$

$$3 = s\left(1 - e^{-2}\right)$$

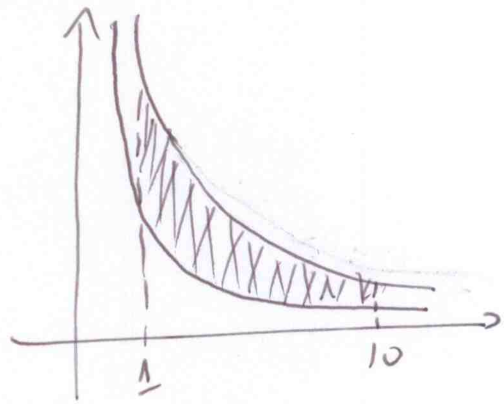
$$\boxed{s = \frac{3}{1-e^{-2}} = \frac{3e^{2}}{e^{2}-1}}$$

10. (10 total points) Consider the region bounded by $x = 1$, $x = 10$, $y = \frac{1}{x}$, and $y = \frac{1}{2x}$.

(a) (8 points) Find the centroid (center of mass) of this region.

$$A = \int_1^{10} \left(\frac{1}{x} - \frac{1}{2x}\right) = \int_1^{10} \frac{1}{2x} dx$$

$$= \frac{1}{2} \ln|x| \Big|_1^{10} = \frac{1}{2} \ln 10.$$



$$\bar{x} = \frac{1}{A} \int_1^{10} x \left(\frac{1}{x} - \frac{1}{2x}\right) dx$$

$$= \left(\frac{2}{\ln 10}\right) \int_1^{10} \left(1 - \frac{1}{2}\right) dx = \left(\frac{2}{\ln 10}\right)\left(\frac{1}{2}x\right)\Big|_1^{10} = \frac{2}{\ln 10} \cdot \frac{9}{2} = \frac{9}{\ln 10}$$

$$\bar{y} = \frac{1}{A} \int_1^{10} \frac{1}{2}\left(\frac{1}{x}\right)^2 - \frac{1}{2}\left(\frac{1}{2x}\right)^2 dx = \left(\frac{\cancel{2}}{\ln 10}\right) \cdot \frac{1}{\cancel{2}} \int_1^{10} \frac{1}{x^2} - \frac{1}{4x^2} dx$$

$$= \frac{1}{\ln 10} \int_1^{10} \frac{3}{4x^2} dx = \frac{3}{4 \ln 10}\left(-\frac{1}{x}\right)\Big|_1^{10} = \frac{3}{4 \ln 10} \cdot \frac{9}{10} = \frac{27}{40 \ln 10}$$

$$\boxed{(\bar{x}, \bar{y}) = \left(\frac{9}{\ln 10}, \frac{27}{40 \ln 10}\right)} \cong (3.91, 0.29)$$

(b) (2 points) Does the centroid lie inside the region? Justify your answer.

$\boxed{\text{No}}$ At $x = \bar{x} \cong 3.91$, the y value of the top function is $y \cong \frac{1}{3.91} \cong 0.256 < \bar{y} \cong 0.29$ so the centroid lies above the region.