1. Evaluate the following integrals. Show your work, and box your final answer.

(a) (7 points) $\int \sin^3(4y)\cos^7(4y)\,dy$

$$= \int \sin^2(4y)\cos^7(4y)\sin(4y)\,dy$$

$$= \int \left(1-\cos^2(4y)\right)\cos^7(4y)\sin(4y)\,dy$$

$u = \cos(4y)$
$du = -\sin(4y)\cdot 4\,dy$

$$= \int (1-u^2)\,u^7\left(-\tfrac{1}{4}\right)du$$

$$= \frac{1}{4}\int u^9 - u^7\,du$$

$$= \frac{1}{4}\left(\frac{u^{10}}{10} - \frac{u^8}{8}\right) + C$$

$$= \boxed{\frac{1}{40}\cos^{10}(4y) - \frac{1}{32}\cos^8(4y) + C}$$

(b) (7 points) $\int \frac{6x^2}{x^2-3x+2}dx$ $= \int 6 + \frac{18x-12}{(x-1)(x-2)}\,dx$

Partial Fractions:

$$\frac{8x-12}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$18x-12 = A(x-2) + B(x-1)$$

$x=1$: $6 = A(-1) \Rightarrow A = -6$

$x=2$: $24 = B(1) \Rightarrow B = 24$

$$= 6x + \int \frac{-6}{x-1} + \frac{24}{x-2}\,dx$$

$$= \boxed{6x - 6\ln|x-1| + 24\ln|x-2| + C}$$

2. (a) (6 points) Evaluate the integral: $\int \sin(\sqrt{x})dx$. Show your work, and box your final answer.

① $u = \sqrt{x} \Rightarrow x = u^2$
$dx = 2u\,du$

$$\int \sin(\sqrt{x})\,dx = \int 2u \sin(u)\,du$$

② Integration By Parts
$w = 2u \quad dv = \sin u\,du$
$dw = 2du \quad v = -\cos u$

$$= -2u\cos u + \int 2\cos u\,du$$

$$= -2u\cos u + 2\sin u + C$$

$$= \boxed{-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C}$$

(b) (6 points) Does the improper integral $\int_0^\infty \frac{1}{\sqrt{2+x^2}}dx$ converge or diverge? Fully justify your answer.

The integral $\boxed{\text{DIVERGES}}$ We can see this via:

1) A COMPARISON TEST: $\sqrt{2+x^2} \le \sqrt{x^2+x^2} = \sqrt{2}x$ for all $x \geq \sqrt{2}$

$$\int_0^\infty \frac{1}{\sqrt{2+x^2}}dx = \underbrace{\int_0^{\sqrt{2}} \frac{1}{\sqrt{2+x^2}}dx}_{\text{positive}} + \int_{\sqrt{2}}^\infty \frac{1}{\sqrt{2+x^2}}dx \geq \int_{\sqrt{2}}^\infty \frac{1}{\sqrt{2}x}dx = \infty$$

OR

2) COMPUTING IT: Trig Sub $x = \sqrt{2}\tan\theta$, $dx = \sqrt{2}\sec^2\theta\,d\theta$.

$$\int_0^{\infty} \frac{1}{\sqrt{2+x^2}}dx = \int_0^{\pi/2} \frac{1}{\cancel{\sqrt{2}}\cancel{\sec\theta}} \cancel{\sqrt{2}}\sec^{\cancel{2}}\theta\,d\theta = \int_0^{\pi/2} \sec\theta\,d\theta$$

$$= \lim_{b\to\pi/2^-} \int_0^b \sec\theta = \lim_{b\to\frac{\pi}{2}^-} \ln|\underset{\downarrow\,\infty}{\sec b} + \underset{\downarrow\,\infty}{\tan b}| = \boxed{\infty}$$

3. The questions on this page are all short answers. No need to justify, except for the very last one.

   (a) Circle the correct equality or inequality signs:

   i. (2 points) $\displaystyle\int_{\pi}^{2\pi} \frac{\sin(x)}{x} dx$ $\quad >\quad = \quad < \quad 0.$

   Integral of a negative function, with increasing bounds

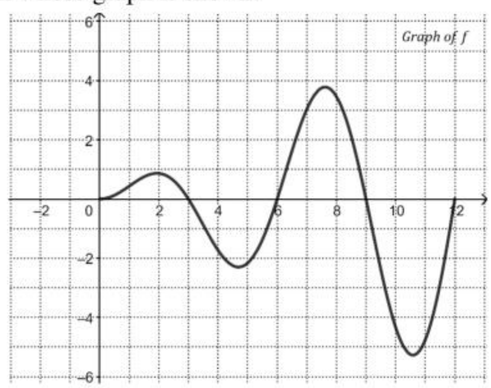
   ii. (2 points) $\displaystyle\int_{\pi}^{\pi} \frac{\sin(x)}{x} dx$ $\quad >\quad = \quad < \quad 0.$

   Integral w/equal bounds.

   (b) The rest of the questions below both refer to a function $g(x)$, which is defined for $0 \le x \le 12$ as:

$$g(x) = \int_0^x f(t)dt,$$

   where $f(t)$ is the function whose graph is shown.



   i. (2 points) At what value of x in the interval $[0,12]$ does $g(x)$ reach its absolute maximum?

   At $\boxed{x=9}$ (compare areas above and below the x-axis)

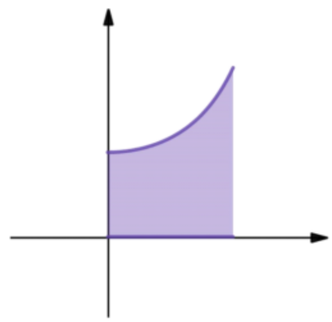
   ii. (4 points) Let $h(x) = g(3x^2 - 5)$. Determine $h'(2)$.

$$h'(x) = g'(3x^2-5)\cdot(6x) \quad \text{Chain Rule}$$
$$= f(3x^2-5)\cdot(6x) \quad \text{FTC I.}$$
$$h'(2) = \underbrace{f(7)\cdot(12)}_{\text{y-value on graph}} = 3\,(12) = \boxed{36}$$

4. (10 points) Find the x-coordinate for the center of mass of a thin plate with constant density that occupies the region:

$$0 \le x \le \pi/4, \qquad 0 \le y \le 2\sec^2(x).$$

Give your answer $\bar{x}$ as an exact number.



$$A = \int_0^{\pi/4} 2\sec^2 x \, dx$$

$$= 2\tan x \Big|_0^{\pi/4} = \boxed{2}$$

$$\bar{x} = \frac{1}{\cancel{A}} \int_0^{\pi/4} x\,(\cancel{2}\sec^2 x)\,dx$$

$$= \int_0^{\pi/4} x \sec^2 x \, dx$$

I.B.P.: $u = x$ $dv = \sec^2 x$; $du = dx$ $v = \tan x$

$$= x\tan x \Big|_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx$$

$$= \left(x\tan x - \ln|\sec x|\right)\Big|_0^{\pi/4}$$

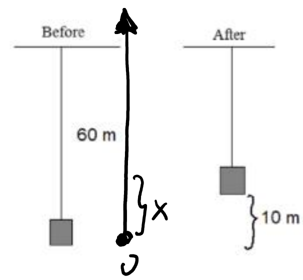
$$= \left(\frac{\pi}{4} - \ln(\sqrt{2})\right) - (0-0)$$

$$= \boxed{\pi/4 - \ln\sqrt{2}} = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

5. (10 points) A 1000-kg elevator is suspended by a 60 m cable that weighs 5 kg/m.

Compute the work, in Joules, required to raise the elevator from the basement to the third floor, a distance of 10 m.

Recall that $g = 9.8$ m/s^2.



The force applied when the elevator is at x meters above its initial position is:

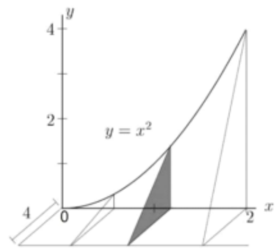
$$F(x) = \text{weight of elevator} + \text{weight of rope}$$
$$= \underset{kg}{1000}\ \underset{m/s^2}{(9.8)} + \underset{kg/m}{5}\ \underbrace{(60-x)}_{m}\ \underset{m/s^2}{(9.8)}$$
$$= 9{,}800 + 49(60-x) \text{ Newtons}$$

$$\boxed{W = \int_0^{10} (12{,}740 - 49x)\, dx} = 12{,}740x - 49\frac{x^2}{2}\Big]_0^{10}$$
$$= 127{,}400 - 49(50)$$
$$= \boxed{124{,}950 \text{ Joules}}$$

Method II:

$$W = W_{(\text{elevator})} + W_{\left(\substack{\text{top 10 m}\\ \text{of rope}}\right)} + W_{\left(\substack{\text{bottom}\\ \text{50 m of rope}}\right)}$$
$$= (9800)(10) + \int_0^{10} (5)(9.8)x\, dx + \int_{10}^{60} (5)(9.8)(10)\, dx.$$
$$= 98{,}000 + 2450 + 24{,}500$$
$$= \boxed{124{,}950 \text{ Joules}}$$

6. (6 points) Compute the volume of the solid shown below. The solid has right triangular cross sections perpendicular to the x-axis, for $0 \le x \le 2$.

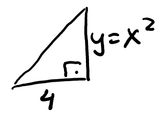


$$V = \int_0^2 A(x)\,dx$$

$$= \int_0^2 \frac{1}{2}(4)(x^2)\,dx$$

$$= \frac{2}{3}x^3 \Big|_0^2$$

$$= \boxed{\frac{16}{3}}$$

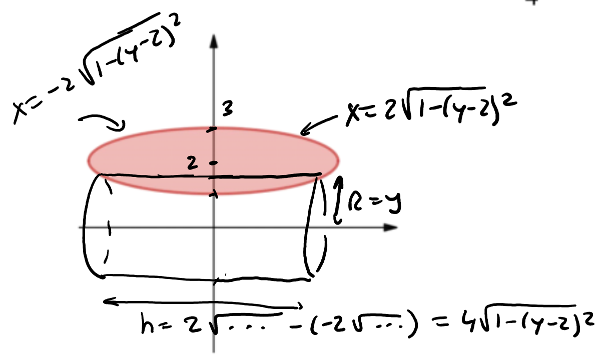


7. (8 points) SET UP only (do not compute) an integral equal to the volume of the solid obtained by rotating about the x-axis the region inside the ellipse:

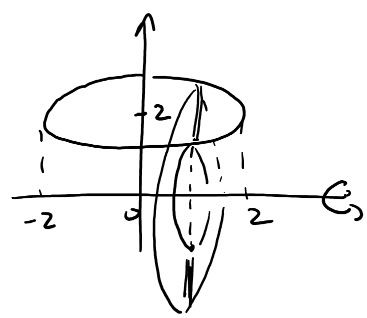
$$\frac{x^2}{4} + (y-2)^2 = 1.$$

$$(y-2)^2 = 1 - \frac{x^2}{4}$$

$$y = 2 \pm \sqrt{1 - x^2/4}.$$



OR



SHELLS:

$$V = \int_1^3 2\pi R(y)\,h(y)\,dy$$

$$= \boxed{\int_1^3 2\pi (y)\left(4\sqrt{1-(y-2)^2}\right)dy}$$

WASHERS:

$$V = \boxed{\int_{-2}^{2} \pi\left(2+\sqrt{1-x^2/4}\right)^2 - \pi\left(2-\sqrt{1-x^2/4}\right)^2 dx}$$

$$= 2\int_0^2 \pi\left(2+\sqrt{1-x^2/4}\right)^2 - \pi\left(2-\sqrt{1-x^2/4}\right)^2 dx.$$

8. (10 points) Find the solution of the differential equation

$$y' + \frac{x}{y-1} = 0$$

that satisfies the initial condition $y(0) = -1$. Give your solution in explicit form, as $y = f(x)$.

$$\frac{dy}{dx} = \frac{-x}{y-1}$$

$$\int (y-1)\,dy = -\int x\,dx$$

$$\frac{y^2}{2} - y = -\frac{x^2}{2} + C$$

$$y(0) = -1: \quad \frac{1}{2} + 1 = -\frac{0^2}{2} + C \implies C = \frac{3}{2}$$

$$\frac{y^2}{2} - y = -\frac{x^2}{2} + \frac{3}{2}$$

$$y^2 - 2y = -x^2 + 3$$

$$y^2 - 2y + (x^2 - 3) = 0$$

Quadratic Formula:

$$y = \frac{2 \pm \sqrt{4 - 4(x^2-3)}}{2} = \frac{2 \pm 2\sqrt{1-(x^2-3)}}{2}$$

$$y = 1 \pm \sqrt{4 - x^2}$$

$$y(0) = -1 \implies \boxed{y = 1 - \sqrt{4-x^2}}$$

9. The change in a population $P = P(t)$ is modeled by the differential equation:

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{5000}\right)$$

Being a population, you may assume the values of P to be non-negative.

(a) (4 points) For what values of P does the population decrease? Justify.

P decreases when $\frac{dP}{dt} = \underbrace{2P}_{\oplus}\left(1 - \frac{P}{5000}\right) < 0$, so when $1 - \frac{P}{5000} < 0$

$\therefore$ for $\boxed{P > 5000}$

(b) (6 points) Which of the following functions are solutions for the given differential equation? Circle all that apply and justify your answer by SHOWING YOUR WORK.

i. $P(t) = 2000$

ii. $P(t) = 5000e^{2t}$

iii. $P(t) = \dfrac{1000}{1+e^{-2t}}$

iv. None of the above

i.) $\frac{dP}{dt} = 0 \neq 2(2000)\left(1 - \frac{2000}{5000}\right) = 2P\left(1 - \frac{P}{5000}\right)$

ii) $\frac{dP}{dt} = 10{,}000e^{2t} \neq 10{,}000e^{2t}\left(1 - e^{2t}\right) = 2P\left(1 - \frac{P}{5000}\right)$

iii) $\frac{dP}{dt} = \frac{e^{-2t}(2000)}{(1+e^{-2t})^2} \neq 2P\left(1 - \frac{P}{5000}\right)$

$$= \frac{2000}{1+e^{-2t}}\left(1 - \frac{1}{5}\frac{1}{1+e^{-2t}}\right)$$

$$= \frac{2000}{(1+e^{-2t})} \frac{5+5e^{-2t}-1}{5(1+e^{-2t})}$$

$$= \frac{2000(4+5e^{-2t})}{5(1+e^{-2t})^2}$$

Alternatively: solve the diff. eq. and show that none of the above match possible solutions

10. (a) (4 points) Set up an integral equal to the arc length of the logarithmic curve

$y = \ln x$, from $(1,0)$ to $(2, \ln 2)$.

$$L = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \boxed{\int_1^2 \sqrt{1+\left(\frac{1}{x}\right)^2}\, dx} \underset{(x>0)}{=} \int_1^2 \frac{\sqrt{x^2+1}}{x}\, dx$$

(b) (6 points) Evaluate the arc length integral from part (a).
(Hint: integration by parts may help in a later step)

TRIG SUB: $u = \tan\theta$
$du = \sec^2\theta\, d\theta$

$\tan\theta = \frac{x}{1}$

$$\int \frac{\sqrt{x^2+1}}{x}\, dx = \int \frac{\sec\theta}{\tan\theta} \sec^2\theta\, d\theta$$

$$= \int \csc\theta \sec^2\theta\, d\theta$$

$$\boxed{\begin{array}{ll} \text{IBP: } u = \csc\theta & dv = \sec^2\theta\, d\theta \\ du = -\csc\theta\cot\theta & v = \tan\theta \end{array}}$$

$$= \underbrace{\csc\theta\tan\theta} - \int -\csc\theta\, \cancel{\cot\theta}\, \cancel{\tan\theta}\, d\theta$$

$$= \sec\theta - \ln|\csc\theta + \cot\theta| + C$$

$$= \sqrt{1+x^2} - \ln\left|\frac{\sqrt{1+x^2}}{x} + \frac{1}{x}\right| + C$$

Evaluating:

$$L = \left[\sqrt{1+x^2} - \ln\left|\frac{\sqrt{1+x^2}+1}{x}\right|\right]\Bigg|_1^2$$

$$= \sqrt{5} - \ln\left(\frac{\sqrt{5}+1}{2}\right) - \left(\sqrt{2} - \ln\left(\frac{\sqrt{2}+1}{1}\right)\right)$$

$$= \boxed{\sqrt{5} - \sqrt{2} - \ln\left(\frac{\sqrt{5}+1}{2}\right) + \ln(\sqrt{2}+1)}$$

$$\cong 1.222$$