1. (14 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) (6 points) $\int_0^{2\pi} \sin^2\theta\, d\theta$ $= \int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\, d\theta$

$$= \frac{1}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right)\Big|_0^{2\pi}$$

$$= \frac{1}{2}\left[(2\pi - 0) - (0-0)\right]$$

$$= \boxed{\pi}$$

(b) (8 points) $\int \frac{1}{x^3+x^2+x}dx$ $= \int \frac{1}{x(x^2+x+1)}\, dx$

Partial Fractions

$$\frac{1}{x(x^2+x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+x+1}$$

$$1 = A(x^2+x+1) + x(Bx+C)$$

$$1 = (A+B)x^2 + (A+C)x + A$$

$\therefore A = 1$

& $A+C=0 \Rightarrow C=-A=-1$

& $A+B=0 \Rightarrow B=-A=-1$.

$$= \int \frac{1}{x} + \frac{-x-1}{x^2+x+1}\, dx$$

$$= \ln|x| - \int \frac{x+1}{(x+\frac{1}{2})^2+\frac{3}{4}}\, dx$$

$u = x+\frac{1}{2}$

$du = dx$

or TRIG SUB

$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$

$dx = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$

$$= \ln|x| - \int \frac{u+1/2}{u^2+3/4}\, du$$

$$= \ln|x| - \int \frac{u}{u^2+3/4}\,du - \frac{1}{2}\int \frac{1}{u^2+(\sqrt{3}/2)^2}\,du$$

$$= \ln|x| - \frac{1}{2}\ln\left|u^2+\frac{3}{4}\right| - \frac{1}{2}\frac{1}{(\sqrt{3}/2)}\arctan\left(\frac{u}{(\sqrt{3}/2)}\right) + C$$

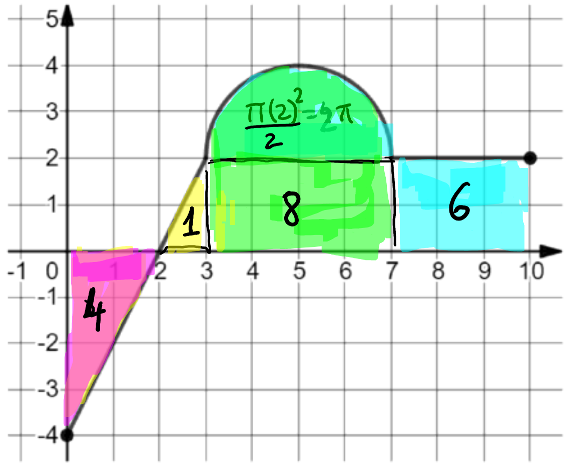
$$= \ln|x| - \frac{1}{2}\ln\left((x+\tfrac{1}{2})^2+\tfrac{3}{4}\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}(x+\tfrac{1}{2})\right) + C$$

$$= \boxed{\ln|x| - \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

Alternative equivalent answer (via trigonometric substitution $u = \frac{\sqrt{3}}{2}\tan\theta$, $du = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$)

$$\boxed{\ln|x| - \ln\left(\frac{2\sqrt{x^2+x+1}}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

2. (10 points) The graph of $y = f(x)$ on $[0, 10]$ is shown below. It consists of two line segments and a half circle. Use it to answer the following questions. Show your steps.




(a) (3 points) Find the average value of $f$ on the interval $[0, 3]$.

$\frac{1}{3}\int_0^3 f(x)\,dx = \frac{1}{3}(-4+1) = \boxed{-1}$

(b) (2 points) Use the graph and areas to compute $\int_0^{10} |f(x)|\,dx$

$= 4+1+8+2\pi+6 = \boxed{19+2\pi}$

(c) (3 points) Define $F(t) = \int_0^t f(x)\,dx$. Evaluate $F'(5)$.

$F'(t) = f(t) \Rightarrow F'(5) = f(5) = \boxed{4}$

(d) (2 points) Evaluate $\int_3^7 \sqrt{1+[f'(x)]^2}\,dx$. *(Hint: There is a quick way to answer this question!)*

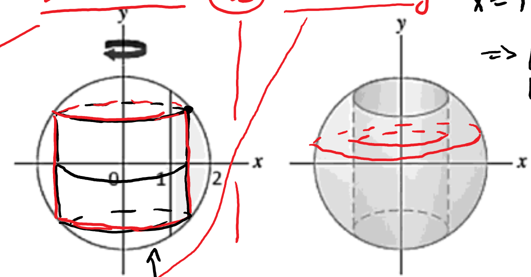
= arclength of $f(x)$ over $[3, 7]$

$= \frac{1}{2}$(perimeter of circle) $= \frac{1}{\cancel{2}}(\cancel{2}\pi R) = \boxed{2\pi}$.

3. (5 points) A "bead" is formed by drilling a hole of radius $r = 1$ cm through the center of a sphere of radius $R = 2$ cm.
Set up an integral equal to the volume of the resulting bead. **Do not compute the integral.**

Shells in x (OR) Washers in y

$x^2+y^2=4$, $x=1$ $\Rightarrow y^2=3$ $\Rightarrow \boxed{y=\pm\sqrt{3}}$ boundary.




$R=\sqrt{4-y^2}$
$r=1$

$x^2+y^2=4$

radius of shell $= x$
height $= \sqrt{4-x^2} - (-\sqrt{4-x^2})$
$= 2\sqrt{4-x^2}$

Shells in x:

$$V = \int_1^2 2\pi\,(\text{radius})\,\text{height}\;dx$$
$$= \boxed{\int_1^2 2\pi x\,(2\sqrt{4-x^2})\,dx}$$
$$= \boxed{4\pi\int_1^2 x\sqrt{4-x^2}\,dx.}$$

OR

Washers in y:

$$V = \boxed{\int_{-\sqrt{3}}^{\sqrt{3}} \pi\left(\sqrt{4-y^2}\right)^2 - \pi(1)^2\,dy}$$
$$= \boxed{\pi\int_{-\sqrt{3}}^{\sqrt{3}} (3-y^2)\,dy}$$

simplified.

4. (5 points) Suppose all we know about some continuous function $g(x)$ is that

$$\frac{1}{x^2} \leq g(x) \text{ for all } x \geq 1.$$

Circle which of the following statements MUST be true, based on the provided information, and justify the statement(s) that you circled.

(a) $\int_1^\infty g(x)\,dx$ diverges

(b) $\int_1^\infty xg(x)\,dx$ diverges

(c) $\int_1^\infty \frac{g(x)}{x}\,dx$ converges

(d) $\int_1^\infty g(x)\,dx$ converges

(e) None of the above

$\frac{1}{x^2} \leq g(x) \Rightarrow x\left(\frac{1}{x^2}\right) = \frac{1}{x} \leq x g(x)$ for all $x \geq 1$.

$\Rightarrow \underbrace{\int_1^\infty \frac{1}{x}\,dx}_{\infty} \leq \int_1^\infty x g(x)\,dx$

$\Rightarrow \int_1^\infty x g(x)\,dx$ diverges

by the Comparison Test.

The others do not give a useful comparison
For ex; in (a), $g(x) \geq \frac{1}{x^2} \Rightarrow \int_1^\infty g(x)\,dx \geq \int_1^\infty \frac{1}{x^2}\,dx = 1$ (finite)
does not allow us to conclude anything about the convergence/divergence of $\int_1^\infty g(x)\,dx$

5. (10 points) A rope is 10 m long, has a total mass of 20 kg, and hangs over the edge of a tall building. Recall that $g = 9.8m/s^2$.

↑ rope weighs $\frac{20\text{kg}}{10\text{m}} = 2$ kg/m.

(a) (5 points) Compute the work required to pull the entire rope to the top of the building. Include **units**.

Let $x$ = length of rope already pulled.
Then the force needed to lift the remaining rope by $\Delta x$ meters is $F(x) \cong 9.8\,(20-2x)$ Newtons
so the incremental work is $9.8(20-2x)\Delta x$ Joules.

Total work:

$$W = \int_0^{10} 9.8(20-2x)\,dx = 9.8\,(20x - x^2)\Big|_0^{10} = 9.8\,(200-100) = \boxed{980 \text{ Joules}}.$$

(b) (5 points) Compute the work required to pull just half the rope to the top of the building.

Depending on method & coordinates used, the integral can be correctly set up in various ways, e.g.:

1) $\int_0^5 9.8(20-2x)\,dx$ with same setup as above

OR

2) $\int_5^{10} 9.8\,(2x)\,dx$ if $x$ = length of rope that's still hanging

OR

3) $\underbrace{\int_0^5 9.8(2x)\,dx}_{\text{top } \frac{1}{2} \text{ of rope}} + \underbrace{9.8(10\text{ kg})(5\text{ m})}_{\text{bottom } \frac{1}{2} \text{ of rope moves up 5 meters}}$

Either way, we get $9.8(75) = \boxed{735 \text{ Joules}}$

6. Consider the region R in the first quadrant bounded by the $y$-axis and by the ellipse

$$x^2 + \frac{(y-2)^2}{4} = 1$$

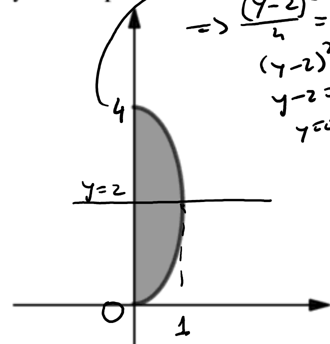
$x = 0$

$\Rightarrow \frac{(y-2)^2}{4} = 1$

$(y-2)^2 = 4$

$y - 2 = \pm 2$

$y = 0$ or $y = 4$




(a) (8 points) Compute the area of this region.

$$A = \int_0^4 \sqrt{1 - \frac{(y-2)^2}{4}}\, dy = \int_0^4 \frac{\sqrt{4-(y-2)^2}}{2}\, dy$$

TRIG SUB: $y - 2 = 2\sin\theta \Rightarrow dy = 2\cos\theta\, d\theta$

$y = 0 \Rightarrow \sin\theta = -1 \Rightarrow \theta = -\pi/2$

$y = 4 \Rightarrow \sin\theta = 1 \Rightarrow \theta = \pi/2$

even function

$$A = \int_{-\pi/2}^{\pi/2} \frac{\sqrt{4 - 4\sin^2\theta}}{\cancel{2}} (\cancel{2}\cos\theta)\, d\theta = \int_{-\pi/2}^{\pi/2} 2\cos^2\theta\, d\theta = 2\int_0^{\pi/2} \cancel{2}\, \frac{1+\cos 2\theta}{\cancel{2}}\, d\theta$$

$$= 2\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2} = (2\theta + \sin 2\theta)\Big|_0^{\pi/2} = \boxed{\pi}$$

$x^2 + \frac{(y-2)^2}{4} = 1$

$(y-2)^2 = 4(1-x^2)$

$y - 2 = \pm 2\sqrt{1-x^2}$

$y = 2 \pm 2\sqrt{1-x^2}$

(b) (8 points) Find the coordinates $(\bar{x}, \bar{y})$ of the centroid of this same region.

By symmetry: $\boxed{\bar{y} = 2}$

$$\bar{x} = \frac{1}{A}\int_0^1 x f(x) - x g(x)\, dx = \frac{1}{\pi}\int_0^1 x(\cancel{2} + 2\sqrt{1-x^2}) - x(\cancel{2} - 2\sqrt{1-x^2})\, dx$$

$$= \frac{1}{\pi}\int_0^1 4x\sqrt{1-x^2}\, dx$$

$\longrightarrow$ $\boxed{\text{u-sub with } u = \sqrt{1-x^2}}$ (or $u = 1 - x^2$)

$u^2 = 1 - x^2$

$\cancel{2}u\,du = -\cancel{2}x\,dx$

$\Rightarrow x\,dx = -u\,du$

$$= \frac{1}{\pi}\int_{u=\sqrt{1-0}=1}^{u=\sqrt{1-1}=0} 4u\,(-u)\, du$$

$$= \frac{1}{\pi}\int_0^1 4u^2\, du$$

$$= \frac{1}{\pi}\,\frac{4}{3}u^3\Big|_0^1 = \frac{1}{\pi}\left(\frac{4}{3}\right) = \frac{4}{3\pi}$$

ANSWER.

$$\boxed{(\bar{x}, \bar{y}) = \left(\frac{4}{3\pi}, 2\right)}$$

7. (8 points) Compute the arc length of the curve

$$y = \frac{e^x + e^{-x}}{2} \implies \frac{dy}{dx} = \frac{e^x - e^{-x}}{2}$$

over the interval $0 \le x \le 1$.

$$L = \int_0^1 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}}\, dx.$$

$$= \int_0^1 \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}}\, dx$$

$$= \int_0^1 \sqrt{\frac{(e^x + e^{-x})^2}{4}}\, dx$$

$$= \int_0^1 \frac{e^x + e^{-x}}{2}\, dx = \frac{1}{2} \int_0^1 e^x + e^{-x}\, dx$$

$$= \frac{1}{2}\left(e^x - e^{-x}\right)\Big|_0^1 = \frac{1}{2}\left[\left(e - \frac{1}{e}\right) - (1 - 1)\right]$$

$$= \boxed{\frac{1}{2}\left(e - \frac{1}{e}\right)} = \boxed{\frac{e^2 - 1}{2e}}$$

8. (10 points) The following two differential equations may appear similar but have very different solutions. Solve both, subject to the same initial condition, as indicated. Show your steps and give your final answers in explicit form, $y = y(x)$.

(a) (5 points) $\frac{dy}{dx} = x$, with $y(1) = 2$.

$$y = \frac{1}{2}x^2 + C$$

$$y(1) = 2 \implies 2 = \frac{1}{2}(1) + C \implies C = \frac{3}{2}$$

$$\therefore \boxed{y = \frac{1}{2}x^2 + \frac{3}{2}}$$

(b) (5 points) $\frac{dy}{dx} = y$, with $y(1) = 2$.

$$\int \frac{1}{y}\,dy = \int dx$$

$$\ln|y| = x + C$$

$$|y| = e^{x+C}$$

$$y = \pm e^{C} e^{x} = A e^{x}$$

$$y(1) = 2: \quad 2 = A e^{1} \implies A = 2/e$$

$$\therefore \boxed{y = \left(\frac{2}{e}\right) e^{x}} \quad \text{i.e.} \quad \boxed{y = 2e^{x-1}}$$

9. (10 points) A container has 75 gallons of liquid in it. At noon, liquid starts being poured into the container at a constant rate of 2 gallons/min. At the same time, liquid starts leaking out of the container through a hole on the bottom, at a rate out that is proportional to the current volume of liquid in the container.

(a) (3 points) Let $V(t)$ denote the volume of liquid in the container, in gallons, at $t$ minutes past noon. Write down the differential equation and initial value satisfied by $V(t)$. Use $k$ for the constant of proportionality.

$$\boxed{\frac{dV}{dt} = \underbrace{2}_{\text{rate in}} - \underbrace{kV}_{\text{rate out}}} \qquad \boxed{V(0) = 75 \text{ gall.}}$$

(b) (7 points) Assume the constant of proportionality is $k = 0.2$. Solve the differential equation from part (a) to obtain an explicit expression for $V = V(t)$, and then determine what happens to the volume of the liquid in the container as time goes on (i.e. as $t \to \infty$).

$$\frac{dV}{dt} = 2 - 0.2V = -0.2(V-10)$$

$$\int \frac{1}{V-10}\, dV = \int -0.2\, dt$$

$$\ln|V-10| = -0.2t + C$$

$$|V-10| = e^{-0.2t+C} = C_1 e^{-0.2t} \qquad (C_1 = e^C)$$

$$V - 10 = C_2 e^{-0.2t} \qquad (C_2 = \pm C_1)$$

$$V(0)=75: \quad 65 = C_2 e^0 = C_2$$

$$\therefore V - 10 = 65e^{-0.2t}$$

$$\boxed{V = 10 + 65e^{-0.2t}}$$

$$\lim_{t\to\infty} V(t) = 10 + 65 \lim_{t\to\infty} e^{-0.2t} = 10 + 65(0) = 10$$

$$\boxed{\lim_{t\to\infty} V(t) = 10 \text{ gallons}}$$

10. Recall that $\int \ln(x)dx = x\ln(x) - x + C$. You may use this without further justification.

(a) (8 points) Compute $\int (\ln x)^2 dx$ and $\int (\ln x)^3 dx$.

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int 2\ln x\, dx$$

$$= x(\ln x)^2 - 2(x\ln x - x) + C$$

$$\boxed{\int (\ln x)^2 dx = x(\ln x)^2 - 2x\ln x + 2x + C}$$

IBP:
$u = (\ln x)^2 \qquad dv = dx$
$du = 2\ln x \cdot \frac{1}{x} \qquad v = x.$

(or $u = \ln x \qquad dv = \ln x\, dx$
$du = \frac{1}{x}dx \qquad v = x\ln x - x$
but this is harder.)

$$\int (\ln x)^3 dx = x(\ln x)^3 - \int 3(\ln x)^2 dx$$

$$= x(\ln x)^3 - 3\left[x(\ln x)^2 - 2x\ln x + 2x\right] + C$$

$$\boxed{\int (\ln x)^3 dx = x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x + C}$$

IBP: $u = (\ln x)^3$ & $dv = dx$
$du = 3(\ln x)^2\frac{1}{x}dx \qquad v = x.$

(Other parts are possible, but harder)

(b) (4 points) Let $k$ be any positive integer. Show that:

$$\int (\ln x)^k dx = x(\ln x)^k - k\int (\ln x)^{k-1} dx$$

IBP: $u = (\ln x)^k$ & $dv = dx$
$du = k(\ln x)^{k-1}\frac{1}{x} \qquad v = x$

$$\int (\ln x)^k dx = x(\ln x)^k - \int k(\ln x)^k \cancel{\frac{1}{x}}\cdot\cancel{x}\, dx$$

$$= x(\ln x)^k - k\int (\ln x)^k dx.$$