1. (10 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\int \cos^3(t) + \tan^2(t)\sec^4(t)\,dt$

$= \int \cos^3 t\,dt + \int \tan^2(t)\sec^4(t)\,dt$

$= \int (1-\sin^2 t)\cos t\,dt + \int \tan^2(t)(1+\tan^2(t))\sec^2(t)\,dt$

$u = \sin t$
$du = \cos t\,dt$

$v = \tan t$
$dv = \sec^2 t\,dt$

$= \int (1-u^2)\,du + \int v^2(1+v^2)\,dv$

$= u - \frac{1}{3}u^3 + \frac{1}{3}v^3 + \frac{1}{5}v^5 + C$

$= \boxed{\sin t - \frac{1}{3}(\sin t)^3 + \frac{1}{3}(\tan t)^3 + \frac{1}{5}(\tan t)^5 + C}$

(b) $\int_e^{e^2} \frac{1}{x(\ln(x))^2}\,dx$

$u = \ln x$
$du = \frac{1}{x}\,dx$

$= \int_1^2 \frac{1}{u^2}\,du$

$= -\frac{1}{u}\Big|_1^2$

$= -\frac{1}{2} + 1 = \boxed{\frac{1}{2}}$

2. (10 points) For each of the following integrals, choose the method that will work best on that integral. No need to justify or to compute anything. Just shade one (and only one) square next to your answer.

(a) $\int \cos^4(x)dx$

- ☐ substitution with $u = \cos^2(x)$
- ☐ replace $\cos^2(x)$ with $1-\sin^2(x)$, then substitution with $u = \sin(x)$
- ■ apply the half-angle formula twice
- ☐ substitution with $u = \cos(x)$

(b) $\int \frac{2x-6}{x^2-6x+12}dx$

- ■ substitution with $u = x^2-6x+12$
- ☐ factor the denominator and use partial fractions
- ☐ integration by parts
- ☐ inverse trig substitution with $x-3 = \sec(\theta)$

(c) $\int \frac{2}{(x^2-1)^{1/2}}dx$

- ☐ substitution with $u = x^2-1$
- ☐ inverse trig substitution with $x = \sin(\theta)$
- ☐ factor the denominator and use partial fractions
- ■ inverse trig substitution with $x = \sec(\theta)$

(d) $\int e^{\sqrt{x}}dx$

- ■ substitution with $u = \sqrt{x}$, then integration by parts
- ☐ integration by parts with $u = e^{\sqrt{x}}$ and $dv = dx$
- ☐ substitution with $u = x$
- ☐ integration by parts with $u = 1$ and $dv = e^{\sqrt{x}}$

(e) $\int \frac{2x}{x^2-9x+2}dx$

- ☐ substitution with $u = x^2-9x+2$
- ☐ integration by parts
- ■ factor the denominator and use partial fractions
- ☐ substitution with $u = x$

3. (a) (6 points) Evaluate $\int x \arctan(x)dx$.

Integration by parts : $u = \arctan x \qquad dv = x\,dx$

$du = \frac{1}{1+x^2}dx \qquad v = \frac{1}{2}x^2$

$$\int x \arctan x \, dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,dx$$

$$= \boxed{\frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C}$$

$$= \frac{1}{2}\left(x^2 \arctan x - x + \arctan x\right) + C$$

$$= \frac{1}{2}\left((x^2+1)\arctan x - x\right) + C_{2*1}$$

(b) (4 points) Is the improper integral $\int_0^\infty x \arctan(x)dx$ convergent or divergent? If it is divergent, justify by showing all limit calculations. If it is convergent, find the value to which it converges.

$$\int_0^\infty x \arctan x \, dx = \lim_{t \to \infty} \left[\frac{1}{2}\left(x^2 \arctan x - x + \arctan x\right)\right]\Big|_0^t$$

$$= \frac{1}{2}\left[\lim_{t\to\infty}\Big(\underbrace{t^2 \arctan t - t}_{\substack{\infty - \infty \\ \text{indeterminate}}} + \underbrace{\arctan t}_{\pi/2}\Big) - (0)\right]$$

$$= \frac{1}{2}\left[\lim_{t\to\infty} \underbrace{t}_{\infty}\,(\underbrace{t\arctan t - 1}_{\infty}) + \frac{\pi}{2}\right] = \boxed{\infty}$$

$\boxed{\text{DIVERGES}}$

4. (10 points) You have the following data about the values of a function $y = f(x)$ and its derivative $\frac{dy}{dx} = f'(x)$ at certain values of x in the interval [0, 2]. Use this table to answer the questions below. Show your setup and your work.

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$f(x)$	1	0.9	0.8	0.5	0	-0.4	-0.7	-0.3	1
$f'(x)$	**0**	**-0.3**	**-1**	**-1.7**	**-2**	**-1.5**	**0**	**3**	**8**

(a) Estimate the average value of f over the given interval $[0, 2]$ using **Simpson's Rule** with $n = 8$ subintervals. Round your answer to the nearest 3 digits.

$$f_{ave} = \frac{1}{2} \int_0^2 f(x)\,dx \qquad \Delta x = \frac{1}{4}$$

$$S_8 = \frac{1}{2} \cdot \frac{1/4}{3} \left[f(0) + 4f(0.25) + 2f(0.5) + \cdots + 4f(1.75) + f(2) \right]$$

$$= \frac{1}{2} \frac{1}{12} \left[1 + 4(0.9) + 2(0.8) + 4(0.5) + 2(0) + 4(-0.4) + 2(-0.7) + 4(-0.3) + 1 \right]$$

$$= \frac{1}{2} \cdot \frac{1}{12} \left[2 + 4(\overbrace{0.9 + 0.5 - 0.4 - 0.3}^{0.7}) + 2(\overbrace{0.8 + 0 - 0.7}^{0.1}) \right] = \frac{1}{2} \frac{1}{12} [2 + 2.8 + 0.2]$$

$$= \frac{1}{2} \frac{1}{12} [5] = \boxed{\frac{5}{24}} \cong \boxed{0.208}.$$

(b) Set up an integral equal to the arc length of $y = f(x)$ over the same interval $[0, 2]$, then estimate that integral using the **Midpoint Rule** with $n = 4$ subintervals. Simplify your answer but leave it in exact form.

$$L = \int_0^2 \sqrt{1 + (f'(x))^2}\,dx \qquad \Delta x = \frac{2}{4} = \frac{1}{2}$$

$$M_4 = \frac{1}{2} \left[\sqrt{1 + (f'(0.25))^2} + \sqrt{1 + (f'(0.75))^2} + \sqrt{1 + (f'(1.25))^2} + \sqrt{1 + (f'(1.75))^2} \right]$$

$$= \frac{1}{2} \left[\sqrt{1 + (-0.3)^2} + \sqrt{1 + (-1.7)^2} + \sqrt{1 + (-1.5)^2} + \sqrt{1 + 3^2} \right]$$

$$= \frac{1}{2} \left[\sqrt{1.09} + \sqrt{3.89} + \sqrt{3.25} + \sqrt{10} \right]$$

$$\cong 3.99$$

5. You have the following data about the values of a function $y = f(x)$ and its derivative $f'(x)$ at certain values of x in the interval [0, 2]. Use this table to answer the questions below.

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$f(x)$	1	0.9	0.8	0.5	0	-0.4	-0.7	-0.3	1
$f'(x)$	0	-0.3	-1	-1.7	-2	-1.5	0	3	8

(a) (3 points) Evaluate $\int_0^{0.75} f'(x)dx$. Do not approximate, find the exact value of this integral.

$$\int_0^{0.75} f'(x)\,dx = f(x)\Big|_0^{0.75} = f(0.75) - f(0) = 0.5 - 1 = \boxed{-0.5}$$

FTC part II

(b) (4 points) Define $F(x) = \int_0^{f(x)} e^{t^2}dt$. Evaluate $F'(0.75)$. Do not approximate, find the exact value of this derivative.

$$F'(x) = \frac{d}{dx}\int_0^{f(x)} e^{t^2}dt = e^{f^2(x)} \cdot f'(x) \quad (\text{FTC I + Chain Rule})$$

$$F'(0.75) = e^{(f(0.75))^2} \cdot f'(0.75) = \boxed{e^{0.25} \cdot (-1.7)} \left(\approx -2.18\right)$$

(c) (3 points) Compute $\int_0^1 xf''(x)dx$. Do not approximate, calculate the exact value of this integral.

$u = x \quad dv = f''(x)\,dx$

$du = dx \quad v = f'(x)$

$$\int_0^1 x f''(x)\,dx = x f'(x)\Big|_0^1 - \int_0^1 f'(x)\,dx = \underbrace{1 f'(1)}_{-2} - \underbrace{f(1)}_{0} + f(0)$$

$$= -2 - 0 + 1$$

$$= \boxed{-1}$$

6. (10 points) A leaky bucket was raised 7 meters. Water was leaking at a constant rate and the bucket was being lifted at a constant speed. At the start, the total mass of the bucket and water was 20 kg. At the top, the mass of the bucket and water was 6 kg. Find the total amount of work that was done.

The mass of the rope was negligibly small. Use $g = 9.8$ m/s^2.

$$\frac{dm}{dx} = \text{constant} = \frac{6\,kg - 20\,kg}{7\,m} = -2\ kg/m$$

The mass of water + bucket when the bucket has been lifted x meters, $0 \le x \le 7$ m is:

$$m = 20 - 2x \quad kg$$

So the force applied to lift the bucket when the bucket is at x meters above ground

$$F = 9.8\,m = 9.8(20-2x) \quad \text{Newtons}$$

Incremental work from x to x + Δx:

$$\Delta W = 9.8\,(20-2x)\,\Delta x \quad \text{Joules}$$

Total work:

$$W = \int_0^7 9.8\,(20-2x)\,dx$$

$$= 9.8\,(20x - x^2)\Big|_0^7$$

$$= 9.8\,(140-49) = 9.8\,(91) = \boxed{891.8 \text{ Joules}}$$

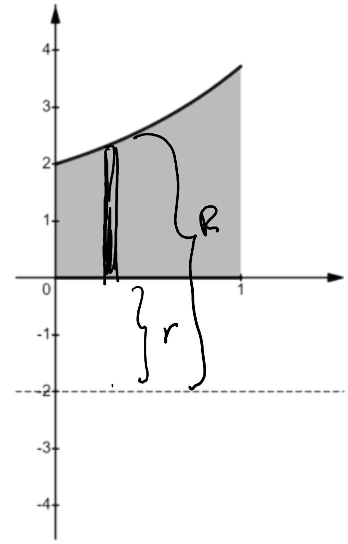
7. (10 points) A solid of revolution is obtained by rotating the region shown below about the horizontal line $y = -2$. The region is above the x-axis, and below the graph of $y = 1 + e^x$, between $x = 0$ and $x = 1$.

(a) Use the method of disks/washers to set up an integral expression equal to the volume of this solid. Do not compute or simplify the integral.

$$V = \int_0^1 \pi R^2 - \pi r^2 \, dx$$

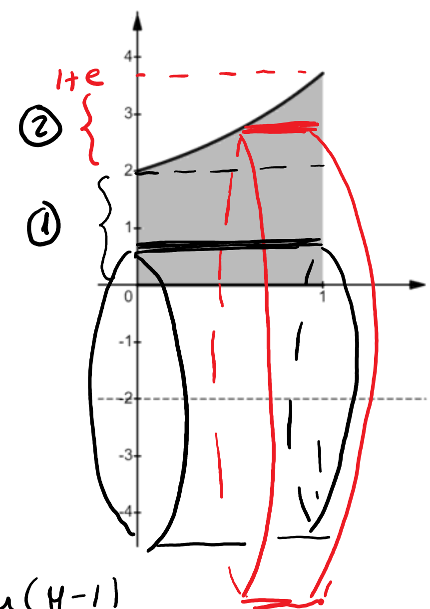
$$= \boxed{\int_0^1 \pi (1+e^x+2)^2 - \pi (2)^2 \, dx}$$

$$= \int_0^1 \pi (e^x+3)^2 - 4\pi \, dx$$



(b) Use the method of cylindrical shells to set up an integral expression for the volume of the same solid. Do not compute or simplify the integral.

$$\boxed{V = \int_0^2 2\pi \overbrace{(y+2)}^{\text{radius}} \overbrace{(1)}^{\text{height}} \, dy + \int_2^{1+e} 2\pi (y+2)(1 - \ln(y-1)) \, dy}$$

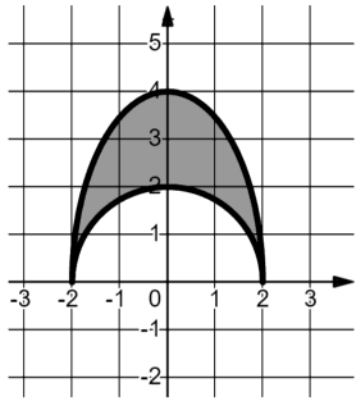


$y = 1 + e^x \Rightarrow e^x = y - 1 \Rightarrow x = \ln(y-1)$

8. (10 points) Find the coordinates $(\bar{x}, \bar{y})$ of the center of mass of a flat plate occupying the shown shaded area in the xy-plane. The region is described by:

$$\sqrt{4-x^2} \le y \le 2\sqrt{4-x^2}$$

You must justify your answers with computations or an explanation.



$$\text{Area} = \int_{-2}^{2} 2\sqrt{4-x^2} - \sqrt{4-x^2}\,dx$$

$$= \int_{-2}^{2} \sqrt{4-x^2}\,dx = 2\pi$$

$\frac{1}{2}$ area disk of radius 2

OR $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$ etc

$\bar{x} = 0$ by symmetry

OR $\bar{x} = \frac{1}{2\pi}\int_{-2}^{2} x\left(2\sqrt{4-x^2}\right) - x\sqrt{4-x^2}\,dx$

$$= \frac{1}{2\pi}\int_{-2}^{2} x\sqrt{4-x^2}\,dx$$

and now use odd function w/ symmetric bounds

OR $u = 4-x^2$ $du = -2x\,dx$

OR $x = 2\sin\theta$ $dx = 2\cos\theta\,d\theta$

$$\bar{y} = \frac{1}{2\pi}\int_{-2}^{2} \frac{1}{2}\left(2\sqrt{4-x^2}\right)^2 - \frac{1}{2}\left(\sqrt{4-x^2}\right)^2 dx$$

$$= \frac{1}{4\pi}\int_{-2}^{2} 4(4-x^2) - (4-x^2)\,dx$$

$$= \frac{3}{4\pi}\int_{-2}^{2} 4-x^2\,dx = \frac{3}{4\pi}\left(2\int_0^2 4-x^2\,dx\right)$$

$$= \frac{3}{2\pi}\left(4x - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{3}{2\pi}\left(8 - \frac{8}{3}\right) = \frac{\cancel{3}}{2\pi}\,\frac{16}{\cancel{3}}$$

$$= \boxed{\frac{8}{\pi}}$$

$$\boxed{\text{Answer: } (\bar{x}, \bar{y}) = \left(0, \frac{8}{\pi}\right)}$$

9. (10 points) Solve the initial value problem. Give your answer in explicit form, $y = f(x)$.

$$\frac{dy}{dx} = \frac{1}{(\tan^2(y)+1)\left(\frac{1}{x}+1\right)} \qquad y(0) = \pi/4$$

$$(\tan^2 y + 1)\frac{dy}{dx} = \frac{1}{\frac{1}{x}+1} = \frac{1}{\left(\frac{x+1}{x}\right)} = \frac{x}{x+1}$$

$$\int (\tan^2 y + 1)\, dy = \int \frac{x}{x+1}\, dx$$

$$\int \sec^2 y \, dy = \int 1 - \frac{1}{x+1}\, dx$$

$$\tan y = x - \ln|x+1| + c$$

$y(0)=\frac{\pi}{4}$: $\quad \tan\frac{\pi}{4} = 0 - \ln 1 + c \Rightarrow c = \tan\frac{\pi}{4} = 1$

$$\tan y = x - \ln|x+1| + 1$$

$$\boxed{y = \arctan(x - \ln|x+1| + 1)}$$

10. A population of mountain goats in a certain region satisfies the differential equation:

$$\frac{dP}{dt} = 3P - \frac{P^2}{1000}$$

where $P = P(t)$ is the number of goats in the region after t years.

(a) (2 points) Find the carrying capacity of the region.

$$\frac{dP}{dt} = 3P\left(1 - \frac{P}{3000}\right) \quad \Rightarrow \quad \underline{K = 3000 \text{ goats}}$$

(b) (8 points) Solve the differential equation and determine the number of goats in the region after 2 years if the initial population was 1000 goats. Show all work.

$$\frac{1000}{3000P - P^2} \frac{dP}{dt} = 1$$

$$\int \frac{1000}{P(3000-P)}\, dP = \int dt$$

$$\int \frac{1/3}{P} + \frac{1/3}{3000-P}\, dP = t + C$$

$$\frac{1}{3}\ln|P| - \frac{1}{3}\ln|3000-P| = t + C$$

$$\ln\left|\frac{P}{3000-P}\right| = 3t + C_1$$

$$\frac{P}{3000-P} = C_2 e^{3t} \qquad (C_2 = e^{C_1})$$

Partial Fractions

$$\frac{1000}{P(3000-P)} = \frac{A}{P} + \frac{B}{3000-P}$$

$$1000 = A(3000-P) + BP$$

$P=0 \Rightarrow A = 1/3$

$P=3000 \Rightarrow B = 1/3$

$t=0,\ P=1000 \Rightarrow C_2 \cdot 1 = \frac{1000}{3000-1000} \Rightarrow C_2 = \frac{1}{2}$

$t=2,\ P=?\ \ \frac{P}{3000-P} = \frac{1}{2}e^6 \Rightarrow P = \frac{1}{2}e^6(3000-P)$

$= 1500e^6 - \frac{1}{2}e^6 P$

$$P = \frac{1500e^6}{(1+\frac{1}{2}e^6)} = \frac{3000e^6}{e^6+2} \approx \boxed{2{,}985} \text{ goats}$$