1. (12 points) Evaluate the following integrals. Box your final answer.

(a) (6 points) $\displaystyle\int \frac{\sin^2(x)\tan(x)}{\sec(x)}\,dx$

$$= \int \frac{\sin^2(x)\,\frac{\sin(x)}{\cos(x)}}{\frac{1}{\cos(x)}}\,dx = \int \sin^3(x)\,dx$$

$$= \int (1-\cos^2(x))\sin(x)\,dx$$

$$\left\{\begin{array}{l} u=\cos x \\ du=-\sin x\,dx \end{array}\right.$$

$$= \int (u^2-1)\,du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \boxed{\frac{1}{3}\cos^3(x) - \cos(x) + C}$$

(b) (6 points) $\displaystyle\int \frac{x^3+2x^2+4x+6}{x^3+3x}\,dx = \int \frac{x^3+3x+2x^2+x+6}{x^3+3x}\,dx$

$$= \int 1 + \frac{2x^2+x+6}{x(x^2+3)}\,dx$$

P.F.

$$\frac{2x^2+x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$2x^2+x+6 = (A+B)x^2+Cx+3A$$

$$\left\{\begin{array}{l} 3A=6 \Rightarrow A=2 \\ C=1 \\ A+B=2 \Rightarrow B=0 \end{array}\right.$$

$$= \int 1 + \frac{2}{x} + \frac{1}{x^3+3}\,dx$$

$$= \boxed{x + 2\ln|x| + \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C}$$

2. (12 points)

Evaluate the following definite integrals. Simplify your answer, but leave it in exact form.

(a) (6 points) $\int_1^4 \sqrt{\sqrt{x}-1}\,dx$

$\boxed{u\text{-sub: } u=\sqrt{x}-1}$ so $\sqrt{x}=u+1$

$x=(u+1)^2$

$\boxed{dx=2(u+1)\,du}$

$$= \int_0^1 \sqrt{u}\cdot 2(u+1)\,du$$

$$= 2\int_0^1 u^{3/2}+u^{1/2}\,du = 2\left(\tfrac{2}{5}u^{5/2}+\tfrac{2}{3}u^{3/2}\right)\Big|_0^1$$

$$= 2\left(\tfrac{2}{5}+\tfrac{2}{3}\right) = \boxed{\tfrac{32}{15}}$$

(b) (6 points) $\int_{-2}^{2} \frac{x^2}{\sqrt{16-x^2}}\,dx$

Trig Sub: $\boxed{\begin{array}{l} x=4\sin\theta \\ dx=4\cos\theta\,d\theta \end{array}}$

Bounds: $x=-2=4\sin\theta \Rightarrow \sin\theta=-\frac{1}{2}$

$\Rightarrow \theta=-\pi/6$

$x=2=4\sin\theta \Rightarrow \sin\theta=\frac{1}{2}$

$\Rightarrow \theta=\pi/6$

$$\int_{-2}^{2}\frac{x^2}{\sqrt{16-x^2}}\,dx = \int_{-\pi/6}^{\pi/6}\frac{16\sin^2(\theta)}{\cancel{\sqrt{16-16\sin^2\theta}}}\,\cancel{4\cos\theta}\,d\theta$$

$$= \int_{-\pi/6}^{\pi/6} 16\sin^2\theta\,d\theta = \int_{-\pi/6}^{\pi/6} 8\left(1-\cos(2\theta)\right)d\theta$$

$$= \left(8\theta-4\sin(2\theta)\right)\Big|_{-\pi/6}^{\pi/6} = \boxed{\tfrac{8\pi}{3}-4\sqrt{3}}$$

3. (10 points) Evaluate the following improper integral showing all the appropriate steps. If the integral diverges, then say so.

$$\int_0^\infty xe^{-2x}\,dx$$

$$= \lim_{t\to\infty} \int_0^t x e^{-2x}\,dx$$

IBP: $u = x \quad dv = e^{-2x}dx$

$du = dx \quad v = -\frac{1}{2}e^{-2x}$

$$= \lim_{t\to\infty} \left(-\frac{1}{2}xe^{-2x}\Big|_0^t - \int_0^t -\frac{1}{2}e^{-2x}\,dx \right)$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2x}\Big|_0^t \right)$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4} \right)$$

$$= -\frac{1}{2}\underbrace{\lim_{t\to\infty} \frac{t}{e^{2t}}}_{\frac{\infty}{\infty},\ \text{L'Hospital}} - \frac{1}{4}\lim_{t\to\infty} e^{-2t} + \frac{1}{4}$$

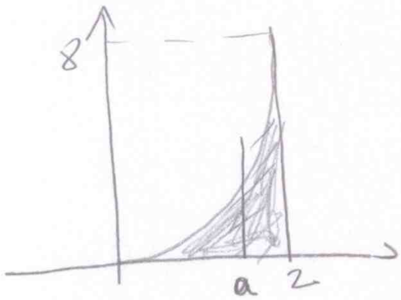
$$= -\frac{1}{2}\lim_{t\to\infty} \frac{1}{2e^{2t}} - \frac{1}{4}(0) + \frac{1}{4}$$

$$= -\frac{1}{2}(0) - \frac{1}{4}(0) + \frac{1}{4}$$

$$= \boxed{\frac{1}{4}}$$

4. (12 points) Consider the region, R, bounded by the curve $y = x^3$, the **vertical** line $x = 2$, and the x-axis.

   (a) (6 points) Find the value of the constant a such that the **vertical** line $x = a$ divides the region R into two regions of equal area.



$$\text{Total Area} = \int_0^2 x^3\,dx = \frac{1}{4}x^4\Big|_0^2 = 4$$

$$\text{Want: } \underbrace{\int_0^a x^3\,dx}_{\frac{1}{4}a^4} = \frac{1}{2}(4) = 2$$

$$\frac{1}{4}a^4 = 2$$

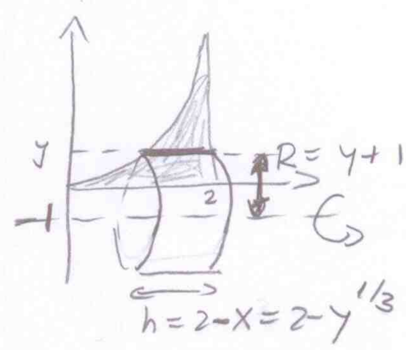
$$a^4 = 8$$

$$\boxed{a = 8^{1/4} = \sqrt[4]{8}}$$

   (b) (6 points) A solid is obtained by rotating the region R around the **horizontal** line $y = -1$. Set up the integrals you get for the volume of this solid using BOTH the method of cylindrical shells and the method of washers (DO NOT EVALUATE).

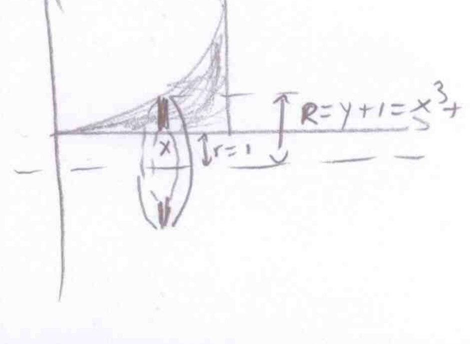
Shells: $\boxed{\int_0^8 2\pi(y+1)(2-y^{1/3})\,dy}$

integral in y



Washers: $\boxed{\int_0^2 \pi(x^3+1)^2 - \pi(1)^2\,dx}$

integral in x



5. (12 points) A cable that weighs 2 lbs per foot is used to lift a bucket of water from a well. The bucket of water weighs 20 lbs, and it needs to be lifted 10 feet to reach the top of the well.

   How far is the bucket from **the bottom of the well** when only half of the total work was done?

Let y denote the height from the bottom of the well.

Then the length of the cable is $10-y$, so the force applied at that height is

$$F(y) = (10-y)\,\cancel{ft}\,(2\,lbs/\cancel{ft}) + 20\ lbs$$

Total work: $W = \int_0^{10} 2(10-y)+20\,dy = \underline{300\ ft\text{-}lbs}$.

Work to lift bucket to b feet from the bottom:

$$\int_0^b 2(10-y)+20\,dy = (40y-y^2)\Big|_0^b = \underline{40b-b^2}$$

Want: $\underline{40b-b^2 = \frac{1}{2}(300) = 150}$

$$b^2-40b+150=0$$

Quadratic Formula: $b = \dfrac{40 \pm \sqrt{1000}}{2} = 20 \pm 5\sqrt{10}$

b must be $\leq 10\,ft$, so $\boxed{b = 20-5\sqrt{10}\ ft} \cong 4.1886\ ft$

6. (10 points) Solve the differential equation:

$$\frac{dy}{dx} = 2\cos^2(x)\cos^2(y) - \cos^2(y)$$

subject to the initial condition

$$y(\pi/4) = \pi/6$$

Give your answer in the form $y = f(x)$.

Separate the variables and integrate:

$$\frac{dy}{dx} = (2\cos^2(x) - 1)\cos^2(y)$$

$$\int \frac{1}{\cos^2(y)}\,dy = \int 2\cos^2(x) - 1\,dx$$

$$\int \sec^2(y)\,dy = \int \cancel{2}\,\frac{1+\cos(2x)}{\cancel{2}} - 1\,dx$$

$$\tan(y) = \frac{1}{2}\sin(2x) + C$$

$y(\pi/4) = \pi/6$: $\tan\left(\frac{\pi}{6}\right) = \frac{1}{2}\sin\left(2\frac{\pi}{4}\right) + C$

$$\frac{1}{\sqrt{3}} = \frac{1}{2} + C \Rightarrow C = \frac{1}{\sqrt{3}} - \frac{1}{2}$$

$$\tan(y) = \frac{1}{2}\sin(2x) + \frac{1}{\sqrt{3}} - \frac{1}{2}$$

$$\boxed{y = \tan^{-1}\left(\frac{1}{2}\sin(2x) + \frac{1}{\sqrt{3}} - \frac{1}{2}\right)}$$

7. (10 points) A particle is moving along a straight line. Its acceleration at t seconds is $a(t) = t$ m/s^2.
The particle begins at the origin, and returns to the origin after 6 seconds.
What is the **total distance** traveled by the particle during that time?

$a(t) = t\ m/s^2$

$v(t) = \frac{1}{2}t^2 + v_0$

$\Delta s = 0 \Rightarrow \int_0^6 v(t)\,dt = 0$

$\left(\frac{1}{6}t^3 + v_0 t\right)\Big|_0^6 = 0$

$\frac{1}{\cancel{6}}6^{\cancel{3}2} + v_0(6) = 0. \Rightarrow v_0 = -6\ m/s$

So: $v(t) = \frac{1}{2}t^2 - 6$

When does the particle turn around?

$v(t) = 0 \Rightarrow \frac{1}{2}t^2 = 6 \Rightarrow t^2 = 12 \Rightarrow t = \sqrt{12} = 2\sqrt{3}$ sec

So the particle starts at the origin, moves in the negative direction for $2\sqrt{3}$ sec, then moves in the positive direction.

Total distance $= \int_0^6 |v(t)|\,dt$

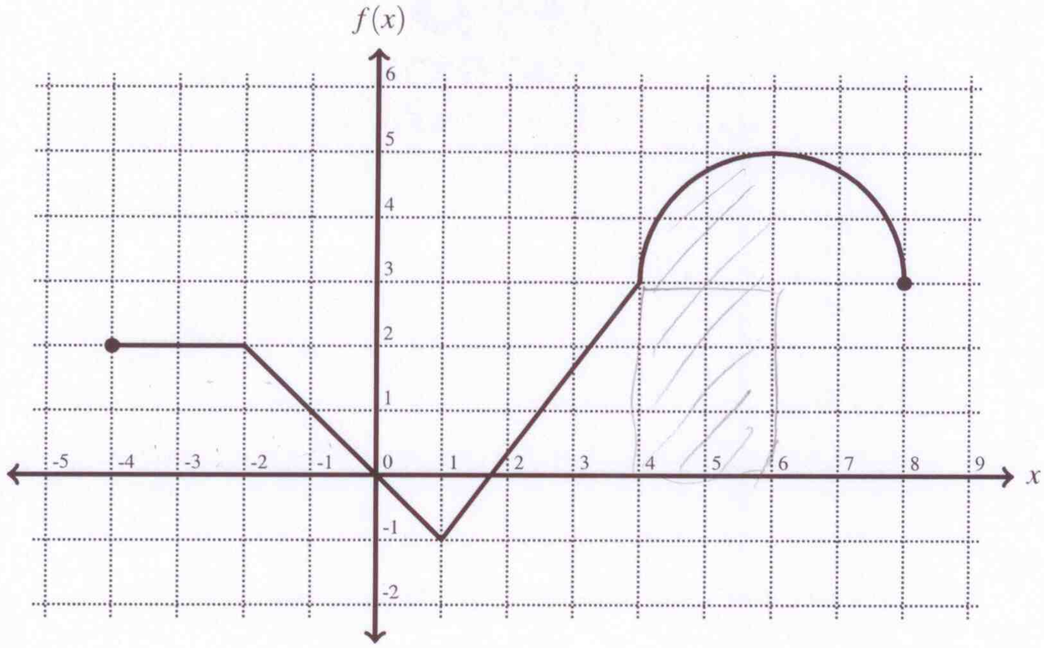
$= \int_0^{2\sqrt{3}} 6 - \frac{1}{2}t^2\,dt + \int_{2\sqrt{3}}^6 \frac{1}{2}t^2 - 6\,dt$

$= \left(6t - \frac{1}{6}t^3\right)\Big|_0^{2\sqrt{3}} + \left(\frac{1}{6}t^3 - 6t\right)\Big|_{2\sqrt{3}}^6$

$= 8\sqrt{3} + 8\sqrt{3}$

$= \boxed{16\sqrt{3} \text{ meters}}$

8. (12 points) The graph of $f(x)$ is shown below. It consists of line segments and a half circle. Use it to answer the following questions.



(a) Compute the average value of $f(x)$ on the interval $[-4,1]$.

$$f_{ave} = \frac{1}{1-(-4)} \int_{-4}^{1} f(x)\,dx = \frac{1}{5}\left(6-\frac{1}{2}\right) = \boxed{1.1}$$

(b) Compute $\int_4^6 \sqrt{1+[f'(x)]^2}\,dx$

(Hint: Think first. You don't need more space than you have here to answer this question!)

This is the arc length of $y=f(x)$ from $x=4$ to $x=6$, which is $\frac{1}{4}$ of the circumference of a circle of radius 2

i.e. $\frac{1}{4}(2\pi(2)) = \boxed{\pi}$

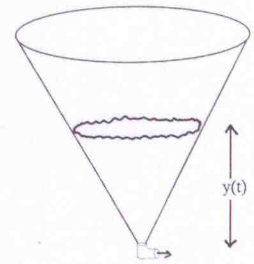
(c) Compute $\int_4^6 x f'(x)\,dx$ $= x f(x)\Big|_4^6 - \int_4^6 f(x)\,dx$ ← [Integration By Parts with $u=x$, $dv=f'(x)dx$, $du=dx$ $v=f(x)$]

$= (6f(6) - 4f(4)) - \left(\text{area under the graph from } x=4 \text{ to } x=6\right)$

$= \underbrace{(6(5)-4(3))} - \left(3\times 2 + \frac{1}{4}\pi(2)^2\right)$

$= 18 - (\pi + 6) = \boxed{12-\pi}$

9. (10 points)



Water is flowing out of a conical container. Let $y(t)$ denote the depth (in feet) of the water in the container and let $V(t)$ denote the volume (in cubic feet) of the water at time t (in minutes). It is known that:

$$(1) \quad \frac{dV}{dt} = -6\sqrt{y}$$

Moreover, V and y are related by the equation:

$$V = 5y^3$$

(a) (2 points) Differentiate the second equation **with respect to time** t.

$$(2) \quad \boxed{\frac{dV}{dt} = 15y^2 \frac{dy}{dt}}$$

(b) (2 points) Use your answer in part (a) to get a differential equation for y.

(1) & $(2) \Rightarrow$ $\boxed{-6\sqrt{y} = 15y^2 \frac{dy}{dt}}$ or, simplified: $\boxed{\frac{dy}{dt} = -\frac{2}{5} y^{-3/2}}$

(c) (6 points) If the initial depth of the water in the container is 4 feet, how long does it take to empty the container?

Separate variables and integrate:

$$\int y^{3/2} dy = \int -\frac{2}{5}\, dt$$

$$\cancel{\tfrac{2}{5}} y^{5/2} = -\cancel{\tfrac{2}{5}} t + C$$

initial depth is 4 ft means at $t=0$, $y=4$ so:

$$4^{5/2} = C \Rightarrow C = 32$$

Hence: $y^{5/2} = -t + 32$

The container is empty when $y=0$, so when

$$0 = -t + 32$$

$$\boxed{t = 32 \text{ minutes}}$$