1. (10 points) Evaluate the following integrals. Show your work, simplify, and box your answers.

(a) $\int \sqrt{x}\ln(x)\,dx$

IBP: $u = \ln x \qquad dv = x^{1/2}dx$

$du = \frac{1}{x}dx \qquad v = \frac{2}{3}x^{3/2}$

$$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int \frac{1}{x}x^{3/2}dx$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}dx$$

$$= \boxed{\frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C}$$

$$= \frac{2}{3}x^{3/2}\left(\ln x - \frac{2}{3}\right) + C$$

(b) $\int \frac{1}{x^2\sqrt{4+x^2}}dx$ Trig. Sub. $x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

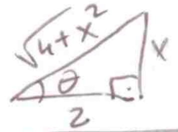
$$= \int \frac{1}{4\tan^2\theta\sqrt{4+4\tan^2\theta}}2\sec^2\theta\,d\theta$$

$$= \int \frac{1}{4\tan^2\theta\,\cancel{2}\cancel{\sec\theta}}\cancel{2}\sec^{\cancel{2}}\theta\,d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}}\frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}d\theta = \frac{1}{4}\int \csc\theta\cot\theta\,d\theta = -\frac{1}{4}\csc\theta + C$$

(OR: $u = \sin\theta \quad du = \cos\theta\,d\theta \Rightarrow \frac{1}{4}\int\frac{1}{u^2}du = \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4\sin\theta} + C$)

Either way:



$\csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp} = \frac{\sqrt{4+x^2}}{x}$

So we get: $\boxed{-\frac{1}{4}\frac{\sqrt{4+x^2}}{x} + C}$

2. (10 points) Evaluate the following integrals. Show your work, simplify, and box your answers.

(a) $\displaystyle\int \frac{x^2-x+2}{x^3+2x}\,dx$

P.F.: $\dfrac{x^2-x+2}{x(x^2+2)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+2}$

$x^2-x+2 = A(x^2+2)+(Bx+C)x$

$x^2-x+2 = (A+B)x^2 + Cx + 2A$

Setting coefficients equal:

$\begin{cases} A+B=1 \Rightarrow B=1-A = 1-1 = 0 \\ C=-1 \\ 2A=2 \Rightarrow A=1 \end{cases}$

$$\int \frac{x^2-x+2}{x(x^2+2)}\,dx = \int \frac{1}{x} + \frac{-1}{x^2+2}\,dx = \boxed{\ln|x| - \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+C}$$

(b) $\displaystyle\int_1^3 \left(\frac{x+1}{x}\right)^2 - |x-2|\,dx$

$$= \int_1^3 \frac{x^2+2x+1}{x^2}\,dx - \int_1^3 |x-2|\,dx$$

$$= \int_1^3 1+\frac{2}{x}+\frac{1}{x^2}\,dx - \left[\int_1^2 2-x\,dx + \int_2^3 x-2\,dx\right]$$

$$= \left(x+2\ln|x|-\frac{1}{x}\right)\Big|_1^3 - \left[\left(2x-\frac{x^2}{2}\right)\Big|_1^2 + \left(\frac{x^2}{2}-2x\right)\Big|_2^3\right]$$

$$= \left(3+2\ln 3-\frac{1}{3}\right)-(1-0-1) - \left[\cancel{2}-\frac{3}{2} + \frac{9}{2}-6-\cancel{2}+4\right]$$

$$= \left(\frac{8}{3}+2\ln 3\right) - [1]$$

$$= \boxed{\frac{5}{3}+2\ln 3}$$

3. (10 points) For each of the two **improper** integrals below, if the integral converges, evaluate it and if it diverges, explain why. Show all limit computations.

(a) $\int_0^3 \frac{x}{3-x}\,dx$

$$\int \frac{x}{3-x}\,dx = \int -1 + \frac{3}{3-x}\,dx = -x - 3\ln|3-x|$$

or : $\displaystyle\int \frac{x}{3-x}\,dx = \int \frac{3-u}{u}(-1)\,du = \int \frac{u-3}{u}\,du = \int 1 - \frac{3}{u}\,du = u - 3\ln|u| + C$

$u = 3-x$
$-du = dx$

$= (3-x) - 3\ln|3-x| + C$

$$\int_0^{3} \frac{x}{3-x}\,dx = \lim_{b\to 3^-} \int_0^b \frac{x}{3-x}\,dx = \lim_{b\to 3^-} \left(-x - 3\ln|3-x|\right)\Big|_0^b$$

$$= \lim_{b\to 3^-} \left(-b - 3\underbrace{\ln\underbrace{|3-b|}_{\downarrow 0^+}}_{\to -\infty} + 0 + 3\ln 3\right)$$

$$= -3 - 3(-\infty) + 0 + 3\ln 3 = \boxed{+\infty} \text{ diverges.}$$

(b) $\int_0^\infty xe^{-3x}\,dx$

$u = x \qquad dv = e^{-3x}dx$
$du = dx \qquad v = -\frac{1}{3}e^{-3x}$

$$\int xe^{-3x}\,dx = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$$

$$\int_0^\infty xe^{-3x}\,dx = \lim_{b\to\infty}\left(\left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right]\Big|_0^b\right)$$

$$= \lim_{b\to\infty}\left(\underbrace{-\frac{1}{3}\frac{b}{e^{3b}}}_{\frac{\infty}{\infty},\ \text{L'Hospital}} - \frac{1}{9}\frac{1}{e^{3b}} + \cancel{\frac{1}{3}(0)e^0} + \frac{1}{9}e^0\right)$$

$$= \underbrace{\lim_{b\to\infty}\left(\left(-\frac{1}{3}\right)\frac{1}{3e^{3b}}\right)}_{=0} - 0 + 0 + \frac{1}{9}\Big) = \boxed{\frac{1}{9}}$$

4. (10 points) Consider the region R bounded by $y = 0$, $x = 1$, $x = 8$, and $y = \frac{16}{x^2}$.

(a) (4 points) Find the area of R.

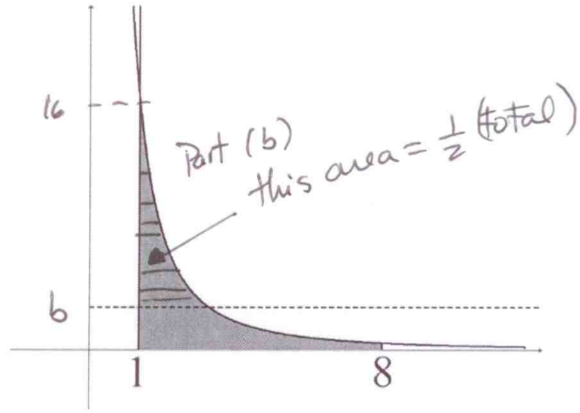
$$A = \int_1^8 \frac{16}{x^2}\,dx$$

$$\left[\text{OR: } \int_0^{16}\left(\frac{4}{\sqrt{y}} - 1\right)dy\right]$$

$$= -\frac{16}{x}\Big|_1^8$$

$$= -2 + 16$$

$$= \boxed{14}$$



(b) (6 points) Find the value of b such that the ***horizontal*** line $y = b$ (the dotted line in the figure above) bisects R into two regions of equal area. You may give your answer either in exact form, or in decimal form accurate to at least two digits after the decimal point.

Easiest setup: $\underbrace{(\text{area above } y=b)} = \underbrace{\tfrac{1}{2}(\text{total area})}$

$$\int_b^{16} \frac{4}{\sqrt{y}} - 1\,dy = \frac{1}{2}(14) = 7$$

$$(8\sqrt{y} - y)\Big|_b^{16} = 7$$

$$(32 - 16) - (8\sqrt{b} - b) = 7$$

$$b - 8\sqrt{b} + 9 = 0$$

Quadratic Formula: $\sqrt{b} = \frac{8 \pm \sqrt{28}}{2} = \frac{8 \pm 2\sqrt{7}}{2} = 4 \pm \sqrt{7}$

If $\sqrt{b} = 4 + \sqrt{7}$ then $b = (4+\sqrt{7})^2 > 16$ too large!

So: $\sqrt{b} = 4 - \sqrt{7}$, i.e. $\boxed{b = (4-\sqrt{7})^2 = 23 - 8\sqrt{7}}$

$\cong 1.834$

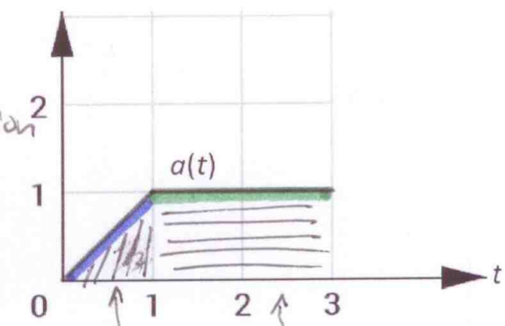
5. (10 points) The acceleration of a particle that moves on a straight line during $0 \le t \le 3$ is given by the graph below.

   Find the **velocity** and the **position** of the particle at time $t = 3$, if the particle started from rest at position $x = 1$ at time $t = 0$.

ACCELERATION:

The main challenge in this question is that all the functions: $a(t)$, $v(t)$, and $s(t)$ are multi-part. For instance, the acceleration function has different expressions on $[0,1]$ and on $[1,3]$:

$$a(t) = \begin{cases} t & \text{if } 0 \le t \le 1 \\ 1 & \text{if } 1 \le t \le 3 \end{cases}$$



VELOCITY:

The easiest way to compute $v(3)$ is to use the Net Change Thm & area:

$$\Delta V = V(3) - V(0) = \int_0^3 a(t)\,dt = \text{area under } a(t) = \frac{1}{2} + 2 = \frac{5}{2}$$

Since $v(0)=0$ (particle started at rest), we get that $\boxed{v(3) = \frac{5}{2}}$.

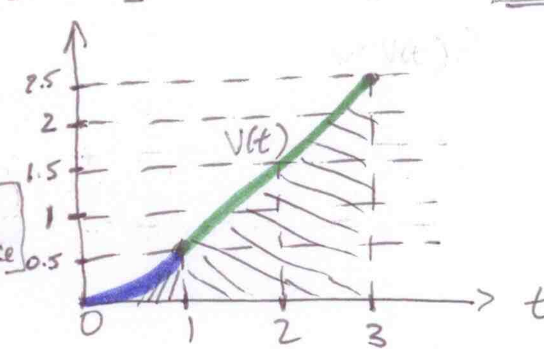
To get the position, though, we'll need to figure out the velocity function, piece-wise, first:

on $[0,1]$: $v(t) = \frac{1}{2}t^2 + C$, $v(0) = 0 \Rightarrow C = 0 \Rightarrow v(t) = \frac{1}{2}t^2$

on $[1,3]$: $v(t) = t + D$ $\quad v(1) = \frac{1}{2}(1)^2 = \frac{1}{2} \Rightarrow 1 + D = \frac{1}{2} \Rightarrow \underline{D = -\frac{1}{2}}$

So: $$v(t) = \begin{cases} \frac{1}{2}t^2 & \text{on } 0 \le t \le 1 \\ t - \frac{1}{2} & \text{on } 1 \le t \le 3 \end{cases}$$

[The 2 "pieces" of $v(t)$ must be equal at $t=1$. This gave us the "initial condition" for the 2nd piece]



POSITION:

$$\Delta X = X(3) - \underbrace{X(0)}_{\text{given}} \overset{\text{Net Change Thm.}}{=} \int_0^3 v(t)\,dt = \int_0^1 \frac{1}{2}t^2\,dt + \int_1^3 \left(t - \frac{1}{2}\right)dt$$

$$X(3) - 1 = \frac{1}{2}\frac{t^3}{3}\Big|_0^1 + \left(\frac{1}{2}t^2 - \frac{1}{2}t\right)\Big|_1^3 = \frac{1}{6} + \left(\frac{9}{2} - \frac{3}{2}\right) - \left(\cancel{\frac{1}{2} - \frac{1}{2}}\right)^{0}$$

$$X(3) - 1 = \frac{1}{6} + 3 \Rightarrow X(3) = \frac{1}{6} + 3 + 1 = \frac{25}{6}$$

$$\therefore \boxed{X(3) = \frac{25}{6}}$$

6. (10 total points) Parts (a) and (b) of this question are unrelated.

   (a) (4 points) Set up (but **DO NOT EVALUATE**) a definite integral equal to the arc length of the curve:

$$y = e^{-x^2}, \qquad 0 \le x \le 3.$$

$$L = \int_0^3 \sqrt{1 + \left[e^{-x^2} \cdot (-2x)\right]^2}\, dx$$

$$= \int_0^3 \sqrt{1 + 4x^2 e^{-2x^2}}\, dx.$$

   (b) (6 points) A CAT scan shows cross-sections (slices) of a patient's liver, spaced 1.5 cm apart. The patient's liver is 15 cm long and the cross-sectional areas, in square centimeters, are:

0, 19, 57, 79, 95, 106, 116, 127, 62, 38, and 0.

Estimate the volume of the liver using the Midpoint Rule with $n = 5$ subintervals.

$$V = \int_0^{15} A(x)\, dx, \qquad \Delta x = \frac{15-0}{5} = 3\text{ cm}$$

$$M_5 = \Delta x\left[A(1.5) + A(4.5) + A(7.5) + A(10.5) + A(14.5)\right]$$

$$= 3\left[19 + 79 + 106 + 127 + 38\right]$$

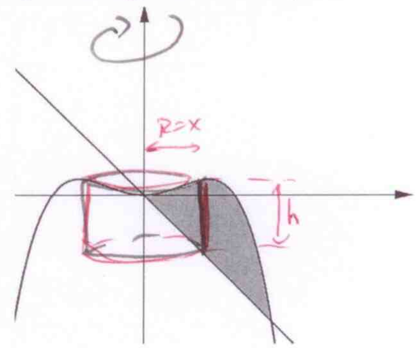
$$= 3\left[369\right] = \boxed{1107\text{ cm}^3}$$

7. (10 total points) Let R be the region bounded by the curves

$$y = x^2 - \frac{1}{2}x^4 \qquad \text{and} \qquad y = -2x.$$

(Note that these curves intersect at $x = 0$ and $x = 2$.)

(a) (6 points) Find the volume of the solid of revolution obtained by rotating R about the y-axis.



Shells:

$$V = \int_0^2 2\pi\, r\, h\, dx$$

$$= \int_0^2 2\pi\, x \left(x^2 - \tfrac{1}{2}x^4 - (-2x)\right)$$

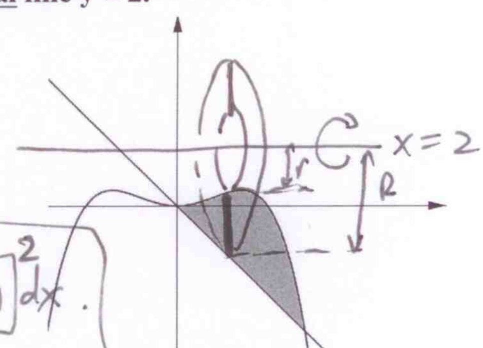
$$= \pi \int_0^2 2x^3 - x^5 + 4x^2\, dx$$

$$= \pi \left[\tfrac{1}{2}x^4 - \tfrac{1}{6}x^6 + \tfrac{4}{3}x^3\right]\Big|_0^2$$

$$= \pi \left[8 - \tfrac{1}{\cancel{3}}\cancel{32} + \tfrac{4}{\cancel{3}}\cdot 8\right]$$

$$= \boxed{8\pi}$$

(b) (4 points) Set up (but **DO NOT EVALUATE** or simplify) a definite integral equal to the volume obtained by rotating the same region R about **the horizontal line y = 2.**



Washers:

$$V = \int_0^2 \pi R^2 - \pi r^2\, dx$$

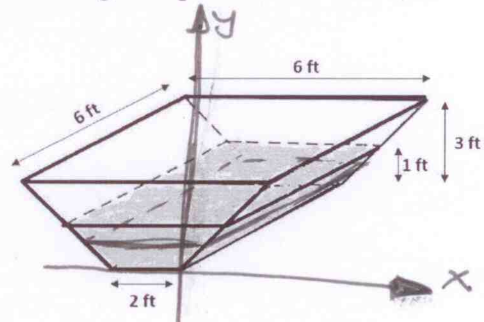
$$= \boxed{\int_0^2 \pi\left[2 - (-2x)\right]^2 - \pi\left[2 - (x^2 - \tfrac{1}{2}x^4)\right]^2 dx.}$$

$$= \boxed{\int_0^2 \pi\left(2 + 2x\right)^2 - \pi\left(2 - x^2 + \tfrac{1}{2}x^4\right)^2 dx}$$

8. (10 points) A water tank is 6 feet long, with two rectangular sides and two trapezoidal ends. Its trapezoidal ends have a height of 3 feet, and bases of 2 feet and 6 feet, as in the picture.

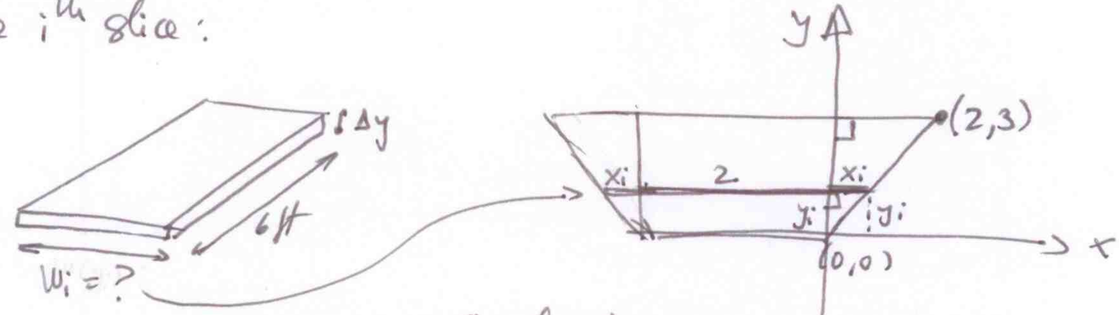
   The tank is filled with water to a level of 1 foot from the bottom. Water weighs 62.5 lb/ft^3.

   SET UP (but **DO NOT EVALUATE** or simplify) a definite integral equal to the work (in ft-lb) required to pump all the water to the top of the tank.



Imposing a coordinate system w/ origin at the bottom, (so y = height above bottom of tank) and "slicing" $0 \le y \le 1$ into n subintervals, we get n horizontal water slices, each of thickness Δy.

The i^{th} slice:



Similar triangles: $\frac{x_i}{2} = \frac{y_i}{3} \Rightarrow x_i = \frac{2}{3} y_i$

OR line $(0,0) \to (2,3)$ has eq. $y = \frac{3}{2}x \Rightarrow x_i = \frac{2}{3} y_i$

$w_i = 2x_i + 2 = 2\left(\frac{2}{3} y_i\right) + 2 = \frac{4}{3} y_i + 2$

Volume $V_i = 6 w_i \Delta y = 6\left(\frac{4}{3} y_i + 2\right)\Delta y = (8y_i + 12)\Delta y$ ft^3

Force $F_i = 62.5 V_i = 62.5(8y_i + 12)\Delta y$

Distance to move slice (to the top): $d_i = 3 - y_i$

Work for the slice: $W_i = 62.5(8y_i + 12)(3 - y_i)\Delta y$

Total Work $= \lim_{n \to \infty} \sum_{i=1}^{n} 62.5(8y_i + 12)(3 - y_i)\Delta y$

$$= \boxed{\int_0^1 62.5(8y + 12)(3 - y)\,dy}$$

9. (10 points) Find the solution to the differential equation

$$y' = \frac{xy^3}{\sqrt{1+x^2}}$$

that satisfies the initial condition

$$y(0) = -1.$$

Give your solution in explicit form, $y = f(x)$.

$$\int \frac{1}{y^3}\,dy = \int \frac{x}{\sqrt{1+x^2}}\,dx \qquad u = 1+x^2 \quad \frac{1}{2}du = \frac{\cancel{2}x}{\cancel{2}}\,dx$$

$$\frac{y^{-2}}{-2} = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$-\frac{1}{2}\frac{1}{y^2} = \frac{1}{\cancel{2}}\left[\frac{u^{1/2}}{\cancel{1/2}}\right] + C$$

$$\frac{1}{y^2} = -2\sqrt{u} + C$$

$$\frac{1}{y^2} = -2\sqrt{1+x^2} + C$$

$$y(0) = -1 : \quad \frac{1}{(-1)^2} = -2\sqrt{1+0^2} + C \Rightarrow 1 = -2 + C \Rightarrow C = 3$$

$$\frac{1}{y^2} = -2\sqrt{1+x^2} + 3$$

$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$$

$$y = \pm\sqrt{\frac{1}{3-2\sqrt{1+x^2}}}$$

$$y(0) = -1 \Rightarrow \boxed{y = -\sqrt{\frac{1}{3-2\sqrt{1+x^2}}}}$$

$$\boxed{y = -\frac{1}{\sqrt{3-2\sqrt{1+x^2}}}}$$

10. (10 total points) A 50-gallon tank initially contains 20 gallons of water in which 10 pounds of salt are dissolved. Pure water enters the tank at a rate of 4 gal/min. Simultaneously, a drain is open at the bottom of the tank, allowing the salt-water solution to leave the tank at a rate of 2 gal/min. The solution is kept thoroughly mixed.

(a) (2 points) Find a formula for the volume $V(t)$ (in gallons) of the salt-water solution in the tank at time t (in minutes).

$$\frac{dV}{dt} = \text{volume rate in} - \text{volume rate out}$$
$$= 4\text{ gal/min} - 2\text{ gal/min} = 2\text{ gal/min}.$$
$$V(t) = 2t + V(0) = \boxed{2t+20} \text{ gallons}.$$

(b) (3 points) Write a differential equation for the amount $y(t)$ (in lbs) of salt in the tank at time t.

$$\frac{dy}{dt} = \left(0\,\frac{\text{lb}}{\text{gal}}\right)\left(4\,\frac{\text{gal}}{\text{min}}\right) - \left(\frac{y\text{ lb}}{2t+20\text{ gal}}\right)\left(2\,\frac{\text{gal}}{\text{min}}\right)$$
$$\boxed{\frac{dy}{dt} = -\frac{y}{t+10}}$$

(c) (3 points) Solve this differential equation and use the initial amount of salt in the tank to find a formula for $y(t)$.

$$\int \frac{1}{y}\,dy = -\int \frac{1}{t+10}\,dt$$
$$\ln|y| = -\ln|t+10| + C$$
$$|y| = e^{-\ln|t+10|} \cdot \boxed{e^C}_{C_1} = C_1 e^{\ln(|t+10|^{-1})}$$
$$|y| = \frac{C_1}{|t+10|} \Rightarrow y = \frac{C_2}{t+10} \quad (C_2 = \pm C_1)$$
$$y(0)=10: \quad 10 = \frac{C_2}{10} \Rightarrow C_2 = 100 \Rightarrow \boxed{y = \frac{100}{t+10}}.$$

(d) (2 points) What is the amount of salt in the tank at the moment that the tank becomes full?

Tank is full when $50 = V(t) = 2t+20 \Rightarrow t = 15$ min

$$y(15) = \frac{100}{15+10} = \boxed{4\text{ lbs}}.$$