1. (10 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\int \tan^5(x)\sec^7(x)\,dx$

$$= \int \tan^4(x)\sec^6(x)\tan(x)\sec(x)\,dx$$

$$= \int (\sec^2(x)-1)^2\sec^6(x)\tan(x)\sec(x)\,dx \qquad \boxed{\begin{array}{l} u = \sec(x) \\ du = \tan x \sec x\,dx \end{array}}$$

$$= \int (u^2-1)^2 u^6\,du = \int (u^4-2u^2+1)u^6\,du$$

$$= \int u^{10} - 2u^8 + u^6\,du$$

$$= \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C$$

$$= \boxed{\frac{1}{11}\sec^{11}(x) - \frac{2}{9}\sec^9(x) + \frac{1}{7}\sec^7(x) + C}$$

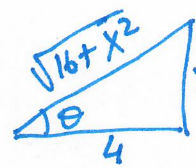
(b) $\int \frac{2}{(x^2+16)^{3/2}}\,dx$

Trig sub $\boxed{\begin{array}{l} x = 4\tan\theta \\ dx = 4\sec^2\theta\,d\theta \end{array}}$

$$= \int \frac{2}{(16(\tan^2\theta+1))^{3/2}}\,4\sec^2\theta\,d\theta$$

$$= \int \frac{2}{4^3\sec^3\theta}\,4\sec^2\theta\,d\theta = \int \frac{1}{8\sec\theta}\,d\theta = \frac{1}{8}\int \cos\theta\,d\theta$$

$$= \frac{1}{8}\sin\theta + C$$



$$= \boxed{\frac{1}{8}\frac{x}{\sqrt{16+x^2}} + C}$$

2. (10 points) Evaluate the following integrals. Show your work. Simplify and box your answers.

(a) $\int_0^1 x\tan^{-1}(x)\,dx$

Integration by parts:

$u = \tan^{-1}x \qquad dv = x\,dx$

$du = \frac{1}{1+x^2}\,dx \qquad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x\Big|_0^1 - \frac{1}{2}\int_0^1 1 - \frac{1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x\Big|_0^1 - \frac{1}{2}x\Big|_0^1 + \frac{1}{2}\tan^{-1}x\Big|_0^1$$

$$= \left(\frac{1}{2}\frac{\pi}{4} - 0\right) - \left(\frac{1}{2} - 0\right) + \left(\frac{1}{2}\frac{\pi}{4} - 0\right)$$

$$= \boxed{\frac{\pi}{4} - \frac{1}{2}} = \boxed{\frac{\pi - 2}{4}}$$

(b) Evaluate the improper integral: $\int_0^\infty \frac{5}{3x^2+6x+6}\,dx = \frac{5}{3}\int_0^\infty \frac{1}{(x+1)^2+1}\,dx$

$$= \frac{5}{3}\int_1^\infty \frac{1}{u^2+1}\,du$$

$\boxed{\begin{matrix} u = x+1 \\ du = dx \end{matrix}}$

$$= \frac{5}{3}\lim_{b\to\infty}\left(\arctan u\Big|_1^b\right)$$

$$= \frac{5}{3}\Big[\underbrace{\lim_{b\to\infty}(\arctan b)} - \arctan 1\Big)$$

$$= \frac{5}{3}\left[\frac{\pi}{2} - \frac{\pi}{4}\right]$$

$$= \boxed{\frac{5\pi}{12}}$$

3. (10 points) A tomato is dropped from the top of a tall building. You do not know when it was dropped, but you see the tomato go by your window, which is 320 feet above the ground. Two seconds after it passes your window, the tomato hits the ground. (The acceleration due to gravity is 32 ft/sec^2.) How tall is the building? Use Calculus methods, and show your work.

Tomato's acceleration (relative to coordinate system w/ origin at ground-level and positive direction upwards) is:

$a(t) = -32$ ft/sec^2

$\therefore$ Velocity $v(t) = -32t + C$

Since the tomato was dropped $v(0)=0$, so $C=0$

$\therefore$ $v(t) = -32t$

Let t_1 be the time when the tomato passes the window. From t_1 to t_1+2, the tomato drops 320 feet (net change in position) so:

$$-320 \text{ ft} = \int_{t_1}^{t_1+2} (-32t)\,dt = -16t^2 \Big|_{t_1}^{t_1+2}$$

$$\therefore \quad -320 = -16\left((t_1+2)^2 - t_1^2\right)$$

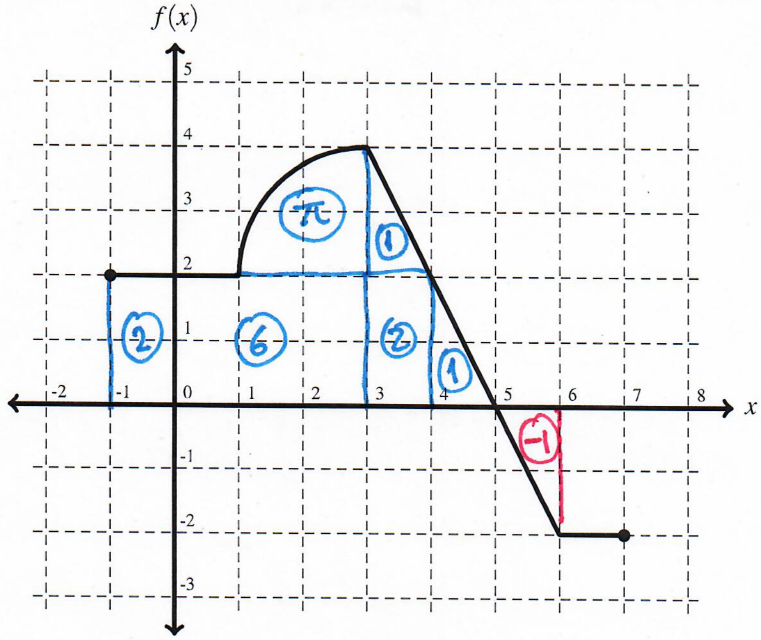
$$\therefore \quad 20 = \cancel{t_1^2} + 4t_1 + 4 - \cancel{t_1^2} \Rightarrow 4t_1 = 16 \Rightarrow \underline{t_1 = 4 \text{ sec}}$$

So, it took 4 sec. after it was dropped for the tomato to pass the window, and 6 sec. to hit the ground. The height h of the building is how much the tomato fell in the $[0,6]$ sec. time interval, so:

$$-h = \int_0^6 -32t\,dt \Rightarrow h = 16t^2 \Big|_0^6 = 16(6)^2$$

$$\boxed{h = 576 \text{ feet}}$$

4. (10 points) The following is the graph of a function $y = f(x)$ over the domain $[-1, 7]$. It consists of line segments and a quarter circle. Use it to answer the questions in parts (a)-(d) below.



(a) Let $g(x) = \int_{-1}^{\sqrt{x}} f(t)\,dt$. Find $g'(9)$.

By FTC I and Chain Rule : $g'(x) = f(\sqrt{x})\cdot(\sqrt{x})' = f(\sqrt{x})\frac{1}{2\sqrt{x}}$

$\therefore g'(9) = f(3)\frac{1}{6} = 4\left(\frac{1}{6}\right) = \boxed{2/3}$

(b) Compute $\lim_{n\to\infty} \sum_{i=1}^{n} \left(f\left(\frac{3i}{n}\right)\frac{6}{n}\right)$ $= \lim_{n\to\infty} \sum_{i=1}^{n} 2f\left(\frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n} 2f(i\Delta x)\Delta x$

$\left(\begin{array}{l} \Delta x = \frac{3}{n} \\ b-a=3\text{; take } [0,3] \\ x_i = 0 + i\Delta x = \frac{3i}{n} \end{array}\right)$

$= \int_0^3 2f(x)\,dx = 2(\text{area under graph over } [0,3])$

$= \boxed{2(6+\pi) = 12+2\pi}$

(c) Evaluate $\int_3^4 x f(x^2-10)\,dx$.

$u = x^2 - 10 \Rightarrow \frac{1}{2}du = x\,dx$ & bounds: $x=3 \Rightarrow u=-1$, $x=4 \Rightarrow u=6$

$= \int_{-1}^{6} \frac{1}{2} f(u)\,du = \frac{1}{2}(\text{signed area over } [-1,6])$

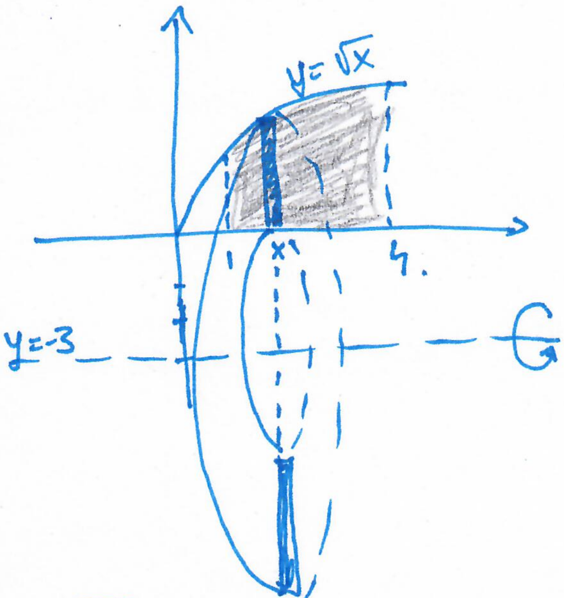
$= \frac{1}{2}(12+\pi-1) = \boxed{\frac{1}{2}(11+\pi)}$

(d) Find the average value of $f(x)$ on the interval $[1,3]$ $= \frac{1}{3-1}\int_1^3 f(x)\,dx$

$= \frac{1}{2}(\text{area over } [1,3]) = \boxed{\frac{1}{2}(4+\pi) = 2+\frac{\pi}{2}}$

5. (10 points) The region in the xy-plane that is above the x-axis, below the curve $y = \sqrt{x}$, and between the lines $x = 1$ and $x = 4$, is rotated around the horizontal line $y = -3$ to form a solid of revolution.

(a) Using washers, set up a definite integral for the volume of this solid, and evaluate this integral to find the volume.



$$V = \int_1^4 \pi R^2 - \pi r^2 \, dx$$

$$= \int_1^4 \pi(\sqrt{x}+3)^2 - \pi(3)^2 \, dx$$

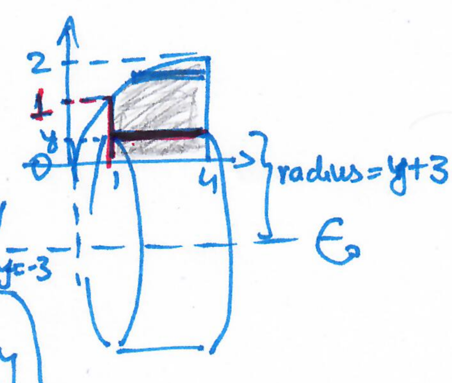
$$= \pi \int_1^4 x + 6\sqrt{x} + \cancel{9} - \cancel{9} \, dx$$

$$= \pi \left(\frac{x^2}{2} + \cancel{6}^{\,2} \frac{2}{\cancel{3}} x^{3/2} \right) \Big|_1^4$$

$$= \pi \left[\left(\frac{16}{2} + 4(\sqrt{4})^3 \right) - \left(\frac{1}{2} + 4 \right) \right]$$

$$= \pi \left[8 + 32 - 4.5 \right] = \boxed{35.5\pi = \frac{71}{2}\pi} \text{ (cubic units)}$$

(b) Using shells, set up definite integral(s) for the volume of this solid, but DO NOT EVALUATE THESE INTEGRAL(S).



$$V = \int_0^1 2\pi(\text{radius})(\text{height})\, dy + \int_1^2 2\pi(\text{radius})(\text{height})\, dy$$

$$= \int_0^1 2\pi\,(y+3)\,(4-1)\, dy + \int_1^2 2\pi\,(y+3)(4-y^2)\, dy$$

$$\therefore \boxed{V = \int_0^1 2\pi(y+3)(3)\, dy + \int_1^2 2\pi(y+3)(4-y^2)\, dy}$$

6. (10 points) This question refers to the arc length of the curve $y = e^{x^2}$ on the interval $[1,7]$.

(a) Set up an integral equal to the arc length of the curve $y = e^{x^2}$ over the interval $[1,7]$. Do not evaluate the integral.

$$L = \int_1^7 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_1^7 \sqrt{1+(2x e^{x^2})^2}\,dx$$

$$= \boxed{\int_1^7 \sqrt{1+4x^2 e^{2x^2}}\,dx}$$

(b) Approximate the arc length of $y = e^{x^2}$ on the interval $[1,7]$ via Simpson's Rule with $n = 6$ subintervals. (Please leave your answer in simplified exact form, rather than writing a decimal.)

Let $f(x) = \sqrt{1+4x^2 e^{2x^2}}$ (integrand above)

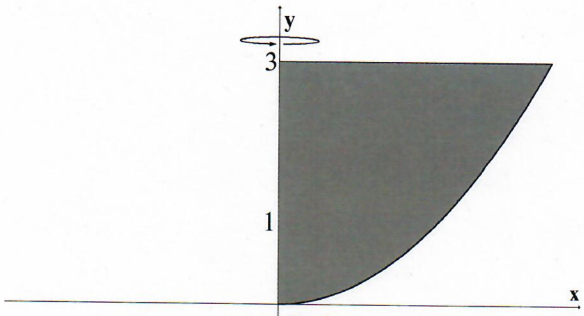
Then

$$L \approx \frac{\Delta x}{3}\left[f(1)+4f(2)+2f(3)+4f(4)+2f(5)+4f(6)+f(7)\right]$$

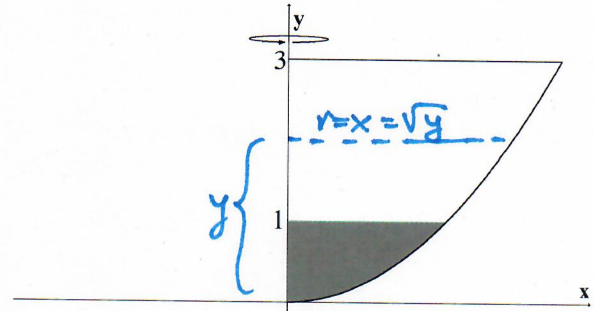
$$= \frac{1}{3}\left[\sqrt{1+4(1)^2 e^{2(1)^2}} + 4\sqrt{1+4(2)^2 e^{2(2)^2}} + \cdots\right]$$

$$= \boxed{\frac{1}{3}\left[\sqrt{1+4e^2} + 4\sqrt{1+16e^8} + 2\sqrt{1+36e^{18}} + 4\sqrt{1+64e^{32}} + 2\sqrt{1+100e^{50}} + 4\sqrt{1+144e^{72}} + \sqrt{1+196e^{98}}\right]}$$

7. (10 points) The region bounded by $y = x^2$, $x = 0$, and $y = 3$ is rotated about the y-axis to form a container. Distances are measured in feet. Initially, the container is full of a fluid that has a density of 50 lbs/ft^3. How much work is done to pump the top 2 feet of liquid to the top of the container?



Before pumping



After pumping

A thin layer of fluid at coordinate y, $1 \le y \le 3$, of thickness Δy would have volume:

$$\cong \pi (r^2) \Delta y = \pi (\sqrt{y})^2 \Delta y = \pi y \Delta y \quad (ft^3)$$

and hence require a force of

$$\cong (\pi y \Delta y)(50) = 50\pi y \Delta y \quad \text{lbs to move}$$

and a work of

$$\cong (\text{force})(\text{dist}) = (50\pi y \Delta y)(3-y) \text{ to lift to top.}$$

Total work for all $1 \le y \le 3$:

$$\boxed{W = \int_1^3 50\pi y(3-y)\,dy}$$

$$= 50\pi \int_1^3 3y - y^2\,dy = 50\pi\left(\frac{3}{2}y^2 - \frac{1}{3}y^3\right)\Big|_1^3$$

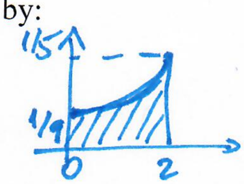
$$= 50\pi\left[\left(\frac{27}{2} - 9\right) - \left(\frac{3}{2} - \frac{1}{3}\right)\right]$$

$$= 50\pi\left[12 - 9 + \frac{1}{3}\right] = 50\pi\,\frac{10}{3} = \boxed{\frac{500\pi}{3} \text{ ft-lbs}}$$

answer

8. (10 points) Find the x-coordinate ($\bar{x}$) of the centroid of the region enclosed by:

$$y = \frac{1}{9-x^2}, \quad x = 0, \quad x = 2, \quad \text{and the } x\text{-axis.}$$



1) Area $A = \int_0^2 \frac{1}{9-x^2}\,dx$

$= \int_0^2 \frac{1/6}{3-x} + \frac{+1/6}{3+x}\,dx$

$\frac{1}{(3-x)(3+x)} = \frac{A}{3-x} + \frac{B}{3+x}$

$1 = A(3+x) + B(3-x)$

$x = 3: \quad A = 1/6$

$x = -3: \quad B = +1/6$

$= \left(-\frac{1}{6}\ln|3-x| + \frac{1}{6}\ln|3+x|\right)\Big|_0^2$

$= -\frac{1}{6}\cancel{\ln 1}^{0} + \frac{1}{6}\ln 5 + \frac{1}{6}\ln 3 - \frac{1}{6}\ln 3$

$\therefore A = \frac{1}{6}\ln 5 \quad (\cong 0.26824)$

2) $\bar{x} = \frac{1}{A}\int_0^2 \frac{x}{9-x^2}\,dx$

$u = 9-x^2$

$du = -2x\,dx$

$= \frac{6}{\ln 5}\int_9^5 \frac{-1/2\,du}{u}$

$= \frac{3}{\ln 5}\int_5^9 \frac{1}{u}\,du$

$= \frac{3}{\ln 5}\ln|u|\Big|_5^9 = \boxed{\frac{3}{\ln 5}(\ln 9 - \ln 5)}$

$= \boxed{\frac{3}{\ln 5}\ln\left(\frac{9}{5}\right)} = \boxed{3\left(\frac{\ln 9}{\ln 5} - 1\right)} \cong 1.0956$

9. (10 points) Find the explicit solution $y = f(x)$ of the initial value problem:

$$\frac{dy}{dx} = x\sqrt{16-y^2}, \qquad y(0) = 2.$$

Separating the variables & integrating, we get:

$$\int \frac{1}{\sqrt{16-y^2}}\,dy = \int x\,dx$$

$$\arcsin\left(\frac{y}{4}\right) = \frac{1}{2}x^2 + C$$

$y(0)=2$: $\arcsin\left(\frac{1}{2}\right) = \frac{1}{2}(0^2)+C \Rightarrow C = \arcsin\frac{1}{2} = \frac{\pi}{6}$

$$\therefore \arcsin\frac{y}{4} = \frac{1}{2}x^2 + \frac{\pi}{6}$$

$$\frac{y}{4} = \sin\left(\frac{1}{2}x^2 + \frac{\pi}{6}\right)$$

$$\boxed{y = 4\sin\left(\frac{1}{2}x^2 + \frac{\pi}{6}\right)}$$

10. (10 points) A type of bacteria grows at a rate proportional to its population, so it is governed by the differential equation $y' = \alpha y$ where y is the number of bacteria after t days and α is constant.

(a) If there are initially 12 bacteria and in 2 days there are 130 bacteria, what is α?

$$\frac{dy}{dt} = \alpha y \Rightarrow \int \frac{1}{y}\,dy = \alpha \int dt$$
$$\Rightarrow \ln|y| = \alpha t + C$$
$$\Rightarrow y = Ae^{\alpha t} \qquad (A = e^C)$$

$$y(0) = 12 \Rightarrow y = 12e^{\alpha t}$$
$$y(2) = 130 \Rightarrow 130 = 12e^{2\alpha} \Rightarrow 2\alpha = \ln\left(\frac{130}{12}\right)$$

$$\boxed{\alpha = \frac{1}{2}\ln\left(\frac{130}{12}\right)} \cong 1.1913139$$

(b) The scientists now (on day 2) start harvesting 100 bacteria every day, so now the equation $y' = \alpha y - 100$ models the situation. If α is as in part (a), how many days of until there are 1,000 bacteria?

$$\frac{dy}{dt} = \alpha y - 100 \Rightarrow \int \frac{1}{\alpha y - 100}\,dy = \int dt \qquad \begin{array}{l} u = \alpha y - 100 \\ du = \alpha\,dy \end{array}$$
$$\Rightarrow \frac{1}{\alpha}\int \frac{1}{u}\,du = t + c$$
$$\Rightarrow \frac{1}{\alpha}\ln|u| = t + c$$
$$\Rightarrow \ln|u| = \alpha t + C_1 \qquad (C_1 = \alpha c)$$
$$\Rightarrow u = \alpha y - 100 = Ae^{\alpha t} \qquad (A = e^{C_1})$$

reset $t=0$ to be day 2 $\Rightarrow y(0) = 130 \Rightarrow \alpha(130) - 100 = A$

$$\therefore \alpha y - 100 = (130\alpha - 100)e^{\alpha t}$$

There are 1000 bacteria when $1000\alpha - 100 = (130\alpha - 100)e^{\alpha t}$

i.e. $t = \frac{1}{\alpha}\ln\left(\frac{1000\alpha - 100}{130\alpha - 100}\right) \cong \boxed{2.51 \text{ days later}}$ (after day 2) so ($\cong 4.51$ days from original time zero).