Improvements and Errata, May 14, 2024

page 29 exercise 2.10: Change the hint to: Consider $e^{\frac{1}{n}f(z)}$ where f is the function in Exercise 2.7(d).

page 51 line 7: "for all $z \in C_r(z_0)$." should be replaced by "for all z inside $C_r(z_0)$."

page 52 line 8: $\gamma_z + \sigma + \partial R$ should be $\gamma_z + \sigma \pm \partial R$.

page 62 problem 4.11: "universal entire function" should have been bold font, and it should be in the index.

page 62 problem 4.12: "Goursat" should have been bold font, and it should be in the index as Goursat's theorem and/or Cauchy-Goursat theorem.

page 64 line 9: $\mathbb{C} \setminus \gamma$ should be $\Omega \setminus \gamma$, though the present form is true in the sense that both integrals are zero if $z \notin \Omega$, by assumption and Theorem 5.1 applied to $f(\zeta)/(\zeta - z)$.

page 68 Definition 5.6: warn the reader that there is equivalent definition using homotopy (see Theorem 14.5). We won't need homotopy until Chapter 14, so there is no need to introduce the concept here.

page 78, 4th line of exercise 5.4: insert "$\gamma'(t) \neq 0$," after the first comma.

page 78: possible additional A exercise (fun to tell the grandkids but maybe it is too easy): Show that you can find your way in and out of any (finite) maze by putting your left hand on the wall at the entrance, then walking forward without taking your hand off the wall. What can you conclude if you exit the maze where you entered?

page 79 line 3: 4.2(b) should be 4.2. (it can be done using either 4.2(a) or 4.2(b)).

page 92 line -6: font for cos is incorrect.

page 101 Figure 6.15: the circle C_3 which is tangent to C_1 should be labeled C_2.

page 101, new problem The Mercator projection $M(w)$ followed by an exponential map gives a conformal map of the punctured sphere onto the plane. The Stereographic projection gives another conformal map $S(w)$ of the punctured sphere onto the plane. Find the Mercator projection in terms of the Stereographic projection and use it to find the integral of the secant without doing any integration. See exercises (1.11) and (6.7). (this exercise requires knowing the exponential and logarithm functions and the conformal maps of the plane onto the plane which fix 0, and an argument that the Mercator projection of the slit sphere extends to a conformal map of the punctured sphere.)

page 108 line 4: $\partial\mathbb{D} \setminus I_\delta$ should be replaced by $[0, 2\pi] \setminus I_\delta$.

page 109 line -8: The limit should be taken over $z \in \mathbb{D}$. Replacing z with $1/\bar{z}$ shows that the limit from outside the disk is $-f(\zeta)$.

page 119 line 16: Move exercise 13.4 to Chapter 7.

page 117: As on page 108, $\partial\mathbb{D}$ is identified with $[0, 2\pi]$ in the proof of Theorem 7.18.

page 130 lines -1,-2,-3: Notational confusion: the g_n in the first two sentences are different from the ones later in the proof. Replace g_n with k_n, $n = 0, 1, 2, ...$ here. If there is a second edition of this book, I'll insert a proposition or theorem in the last section of Chapter 6 which says that a simply-connected region contained in, but not equal to, $\mathbb{C}$ can be conformally mapped into $\mathbb{D}$, and remark that later we'll prove that the map can be taken to be onto $\mathbb{D}$.

page 132 line 10: "Replace the sentence "The set U is non-empty if δ is sufficiently small." By "The region U is simply-connected, for if $\gamma \subset U$ is a closed curve with $n(\gamma, \zeta_1) \neq 0$, for some $\zeta_1 \in \Omega \setminus U$, then ζ_1 belongs to a square S_1 such that one of the 9 squares, $S_2 \subset 3S_1$, contains a $\zeta_2 \notin \Omega$. Then $\overline{S_1 \cup S_2}$ is connected and does not intersect γ. Thus $n(\gamma, \zeta_2) = n(\gamma, \zeta_1) \neq 0$, contradicting Theorem 5.7 for Ω.

page 132 line 17: $8\delta_n$ should be $\delta_n/8$

page 132 line 19: Ω_n should be Ω_{δ_n} in two places.

page 132 line -15: In the statement of Theorem 8.13, change "Suppose Ω" to "Suppose $\Omega \subsetneq \mathbb{C}$". Then delete $\neq \mathbb{C}$ on line -14.

page 135 line 13: Theorem 8.15 should be Corollary 8.18

page 155 Add an exercise outlining $\mathcal{L}(f) = \mathcal{L}(g)$ implies $f = g$, under suitable hypotheses.

page 156 line 17 and page 157 line 6: $\Omega = \bigcup_{j=1}^{\infty} \Delta_j^\circ$, where Δ_j° is the interior of a closed disk $\Delta_j \subset \Omega$.

page 160 line -1 $4M/r^2$ should be $4M/r$.

page 166 line 9 change ", uniformly" to "which is uniformly convergent"

page 167 line 14: "There exist K, $K > 0$, such that, if" should be "There exists K, $K > 0$, such that if"

page 169 exercise 10.10: n is a positive integer.

page 169 exercise 10.11: c is supposed to be a constant which may depend on f.

page 172 line -4: convergence is uniform on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$ (not $\mathbb{C}$)

pages 174, lines 2-8: Can replace "However....Because $\sum 2R/n^2$," with "By Proposition 11.2, with $n = 0$,"

page 174 last two lines: can be replaced by the explicit estimate $|mw_1 + nw_2| \geq |w_2|\ |\text{Im}(w_1/w_2)|$, if $m \neq 0$, and $\geq |w_2|$ if $m = 0$ and $n \neq 0$.

page 175, fourth display $\zeta_{m,n\neq 0}$ should be $\zeta_{m,n} \neq 0$.

page 178 line 7: "in $\mathbb{C} \setminus K_n$" should be "on K_n".

page 186 line -5: "increases" should be "decreases". The picture suggests considering the argument principle, tracing the boundary of a large half disk in Re$z \geq 1/2$ counter-clockwise. The one errant loop is due to the pole at $z = 1$.

page 188 line 11: "We proved in Theorem 11.23" (instead of 11.24).

page 196, Figure 12.2: Someone with sharp eyes pointed out that the curve has a tiny gap.

page 202 line 7: "Section 8.1" should be "the first paragraph of Chapter 8."

page 203 lines 8-15: A better wording of this paragraph would be to insert lines 13-15 after "is bounded" on line 8. Then replace "Because $\varphi(\mathbb{D}) \cap V = \emptyset$ and $\sigma_n \subset \partial\Omega$," with "But $\gamma_z \cap (\sigma_n \cup \varphi(\gamma_{\delta_n})) = \emptyset$, because $V \cap \sigma_n = \emptyset$ and $V \cap \varphi(\mathbb{D}) = \emptyset$, so"

page 206 Exercise 12.8(b): Hint: Apply the argument in (12.1) using subsets of circles centered at one end point of σ_n, but integrate between $d_n = \text{diam}\sigma_n$ and $\sqrt{d_n}$ to show diam$\psi(\sigma_n) \to 0$.

page 209 line 1: Definition 13.1(iii) (not 1.13)

page 209 line -5: "region is regular" should be "region except $\mathbb{C}$ is regular."

page 210 line 1: "C" should be "γ"

page 210 line -2: "Given $0 < \delta < \varepsilon$" should be replaced by "Given $\delta > 0$"

page 213 line 21: "$\partial\Omega = \partial\mathbb{D} \cup E_1$, where $E_1 \subset \mathbb{D}$" should be "$\partial\Omega = \partial\mathbb{D} \cup \partial E_1$, where $\partial E_1 \subset \mathbb{D}$"

page 214 Exercise 13.5: u_f is bounded because f is bounded on $\partial\Omega$, so Green's function exists.

page 215, line 3: residue = -1

page 215 Exercise 13.6: put the Hints at the end of the problem.

page 217 lines 17 and 22: Ref should be Ref_0

page 219 line -1: 0 should be b.

page 221, lines -16, -15: analytic, Harmonic, subharmonic, meromorphic should be bold font, and referenced in the index.

page 221 line -4: $f' \circ z_\alpha^{-1}(z)/g' \circ z_\alpha^{-1}(z)$ should be $(f \circ z_\alpha^{-1})'/(g \circ z_\alpha^{-1})'$

page 227 Exercise 14.3: reword: Define a **homotopy** in a Riemann surface Ω. Then prove the following for curves contained in Ω.

page 228, line -7: change " is a disk" to "is simply-connected".

page 229, Exercise 14.12: "Riemann surface V" should be "simply-connected Riemann surface V". Also it would be good to add a part (b) to this exercise: If W_1, W are Riemann surfaces and if $\pi_1 : W_1 \to W$ evenly covers W, then there exists $g : W^* \to W_1$ which evenly covers W_1, where W^* is the universal cover of W. This is the origin of the name "universal cover".

page 231 line -7: "see" should be "seem"

page 232 line -12: The condition $K \neq W$ should be removed.

page 233 line 3: Replace with: "On W. Note that a Green's function does not exist on a compact surface because all constant functions are in the Perron family."

page 233 line 5: after "Theorem 7.15" add: "and exercise 15.1"

page 233 lines -9 and -2 and page 234 line 14: After "Lindelöf's maximum principle" add "on $U \setminus \{p_0\}$".

page 233 line -3: $z_\alpha(U)$ should be $z(U)$.

page 235 line 15: $\mathcal{F}$ should be $\mathcal{F}_{p_0}$

page 235 after line 16: Add: One consequence of Theorem 15.5 is a (non-local) version of Lindelöf's maximum principle on a Rieman surface. See exercise 15.1. (we'll add a new exercise 15.1, see below, p. 243)

page 236 line 4: replace "on the interior of $K^* \setminus \{q_1^*, \ldots, q_n^*\}$," with "see exercise 15.1". (this refers to the new exercise 15.1, see above.

page 236 last paragraph of the proof of Theorem 15.6: Replace this paragraph with the contents of the last page of this document.

page 238 lines 2, 3: replace "applied on the interior of $K \setminus \{p_0\}$" with: (see exercise 15.1),"

page 238 line 13: replace "difference of two analytic functions with the same real part" with "ratio of two analytic functions with the same absolute value"

page 241 display (15.22): A better name for $M_1(t)$ is $M_r(t)$.

page 241, line 12: after "Harnack's inequality" insert ", Corollary 7.19,"

page 243 line -6: "show that" should be "and show that".

page 243 line -1: "then e^z" should be "then use e^z"

page 243 line -12: Insert a new exercise and renumber all the others (and all references to them):

15.1 Suppose W is a Riemann surface and suppose K is a compact subset of W with $W \setminus K \neq \emptyset$. If $E = \{\zeta_1, \ldots, \zeta_n\} \subset K$ and if v is subharmonic on $W \setminus E$ such that $v \leq M < \infty$ and $v \leq m$ on $W \setminus K$ then $v \leq m$ on W. This is a version of Lindelöf's maximum principle on a Riemann surface. Hint: replace $\log|(z-\zeta_j)/d|$ in the proof of Theorem 7.15 with $-g_{W\setminus\overline{U_0}}(z,\zeta_j)$ where U_0 is a coordinate disk with compact closure contained in $W \setminus K$ (see Theorem 15.5).

page 247 line 8: "For the remainder of this chapter" should be "For the remainder of this section and section 16.3"

page 249 line 1: change "an elliptic"to "a nonconstant elliptic"

page 249 line 6: change "elliptic" to "elliptic and nonconstant"

page 249 line -3: insert "about 0" after winding number.

page 253 line 5: "$\mathbb{D} \setminus \mathbb{G}$" should be $\mathbb{D}/\mathbb{G}$"

page 256 line 21: "expression for $F(z)$" should be "expression for $-F(z)$"

page 257 line 3: Add: A point $w \in W$ is a critical point of g_W if $\nabla(g_W \circ z_\alpha^{-1}) = ((g \circ z_\alpha^{-1})_x, (g \circ z_\alpha^{-1})_y) = (0,0)$ at $z_\alpha(w)$, where z_α is a chart map defined in a neighborhood of w.

page 257 line -9 "in Section 16.3" should be "in section 16.4".

page 260 line 13: item 5 should say "closed curves". Also we could cite Corollary 4.18 and Theorem 4.32, along with Morera.

page 261: Add 12th item to the Exercises: Ω is homeomorphic to the unit disk.

page 269 line -8: under curve, add: analytic, 125, 138

page 269 line -1: under Cauchy's theorem add: Goursat's proof, 62

page 270 line -4: insert: Goursat's proof of Cauchy's theorem, 62

page 270: change "homotopy 218" to "homotopy 218, 227"

page 271: under Lindelöf's maximum principle, 115 add: "on a Riemann surface, 243" (see above for update to page 243).

page 272: subharmonic, 106 should be subharmonic, 105.

back cover: perhaps I should add: Each loop is a hyperbolic geodesic in the complement of the blue curve.

possible addition for new edition: Add a new section in chapter 16 proving some version of the Belyi/Grothendieck theorem and dessin d'enfants. This would be a good example of building non-trivial Riemann surfaces.

If you find errors or have suggestions for improvement, please send them to me at dmarshal@uw.edu.

Thanks to all who have contributed their observations. Don Marshall

Details page 236 replacing the last paragraph in the proof of Theorem 15.6:

We claim that if $\pi(p_1^*) = \pi(p_2^*)$ then $S(p_1^*) = S(p_2^*)$ and hence $S \circ \pi^{-1}$ is a well-defined, non-negative, harmonic function on $W \setminus p_0$. Assuming the claim for the moment, suppose $v \in \mathcal{F}_{p_0}$, the Perron family used to construct $g_W(p, p_0)$. By (15.9) and (15.1b), $v - S \circ \pi^{-1}$ is subharmonic on $W \setminus p_0$, bounded on K, and non-positive on $W \setminus K$, where K is the compact set in (15.1a). By Lindelöf's Maximum principle $v - S \circ \pi^{-1} \leq 0$ on $W \setminus p_0$. Taking the supremum over all such v, then composing with π, we conclude $g_W(\pi(p^*), p_0) \leq S(p^*)$ on W^*.

To see the claim, if σ is a deck transformation of W^* then σ is a one-to-one analytic map of W^* onto W^* by Corollary 14.15. Thus $v \in \mathcal{F}_{q*}$ if and only if $v \circ \sigma \in \mathcal{F}_{\sigma^{-1}(q^*)}$, and hence $g_{W^*}(\sigma(p^*), q^*) = g_{W^*}(p^*, \sigma^{-1}(q^*))$. Choose a deck transformation σ so that $\sigma(p_1^*) = p_2^*$. If $\pi(q_j^*) = p_0$ and $v_j \in \mathcal{F}_{q_j^*}$, $1 \leq j \leq n$, then

$$\sum_1^n v_j \circ \sigma(p^*) \leq \sum_1^n g_{W^*}(p*, \sigma^{-1}(q_j^*)).$$

Taking the supremum over all such v_j and n we conclude

$$S(\sigma(p^*)) \leq S(p^*).$$

Replacing p^* with $\sigma^{-1}(p^*)$ we conclude $S(\sigma(p^*)) = S(p^*)$, proving the claim.