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OBLIQUELY REFLECTED BROWNIAN MOTION IN NON-SMOOTH PLANAR DOMAINS

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We construct obliquely reflected Brownian motions in all bounded simply connected planar domains, including non-smooth domains, with general reflection vector fields on the boundary. Conformal mappings and excursion theory are our main technical tools. A key intermediate step, which may be of independent interest, is an alternative characterization of reflected Brownian motions in smooth bounded planar domains with a given field of angles of oblique reflection on the boundary in terms of a pair of quantities, namely an integrable positive harmonic function, which represents the stationary distribution of the process, and a real number that represents, in a suitable sense, the asymptotic rate of rotation of the process around a reference point in the domain. Furthermore, we also show that any obliquely reflected Brownian motion in a simply connected Jordan domain can be obtained as a suitable limit of obliquely reflected Brownian motions in smooth domains.

1. Introduction. Obliquely reflected Brownian motion (ORBM) arises naturally in some applied probabilistic models, for example, in queuing theory; see Ramanan (2006); Williams (1998) and the references therein. This part of the theory of ORBMs is mostly concerned with processes confined to the positive orthant of the Euclidean space with constant reflection direction on each face. ORBMs in non-smooth (fractal) domains serve as a toy model for some biological phenomena (see Hołyst et al. (2000)). In this paper, we will construct and investigate ORBMs in bounded simply connected planar domains, including non-smooth domains, with variable and possibly non-smooth reflection directions. Conformal mappings will be our main technical tool. The construction of ORBM in a general non-smooth domain is difficult because the process (if it exists) is non-symmetric and, therefore, the (symmetric) Dirichlet form approach (see Fukushima (1967); Chen (1993) and the references therein), very successful in the case of normally reflected Brownian motion, is not applicable to ORBM with general non-smooth reflection directions.

A conceptual problem with obliquely reflected Brownian motion is that the oblique reflection represents, in heuristic terms, a slight push away from the boundary accompanied by a proportional push along the boundary. In fractal domains, the concepts of "normal" direction at a boundary point and moving "along" the boundary do not have a meaning according to classical definitions. Hence describing and classifying ORBMs in such non-smooth domains

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requires a new approach. The key to our study is the observation that ORBMs in smooth domains can be fully and uniquely classified using two "parameters"—an integrable positive harmonic function h and a real number μ_0. The harmonic function h represents the density of the stationary distribution of the process and the real number μ_0 represents, in an appropriate sense, the asymptotic rate of rotation around a reference point in the domain. This alternative characterization of ORBM will allow us to construct and investigate ORBM in non-smooth planar domains with general reflection on the boundary. More specifically, we will first show in Theorems 3.2 and 3.5 that h and μ_0 provide a parametrization of ORBMs in the unit disc alternative to the reflection vector field on the boundary. Then we will show in Theorems 3.17-3.19 how ORBMs in non-smooth domains can be constructed and classified.

Yet another "parametrization" of ORBM's in simply connected domains is given by "rotation rates" $\mu(z)$ of the process around points z in the domain. Every function $\mu(z)$ representing rotation rates is harmonic but not every harmonic function $\mu(z)$ represents rotation rates for an ORBM.

We will also discuss some ORBMs with degenerate ("tangential") "reflection" along the boundary. The infinitely strong tangential push generates jumps along the boundary, a feature not normally associated with models labeled "Brownian." We will show that ORBMs with "degenerate" boundary behavior are processes that recently appeared in the probabilistic literature in a different context.

The present paper can be viewed as a first step in a much more ambitious project to define ORBMs in d-dimensional non-smooth domains with $d \geq 2$. In the two-dimensional case, especially in simply connected domains, one can give a meaning to the "angle of reflection" even in domains with fractal boundary by approximating the boundary with continuous curves, defining the angle of reflection on these curves, then defining the corresponding ORBMs and finally passing to the limit (see Theorem 3.19 below). The same program is questionable in higher-dimensional domains. It is not clear how to define the direction of reflection on a fractal boundary or how to define the direction of reflection on a sequence of approximating smooth surfaces in a "consistent" way. We believe that our approach via the stationary density (see Kang and Ramanan (2014) for a characterization of stationary distributions of ORBMs in d-dimenesional piecewise smooth domains) and appropriate "rotations about $(d-2)$-dimensional sets" may be the right approach to the high-dimensional version of the problem but we leave it for a future project.

There are two classes of domains to which some of our results should extend in a fairly straightforward way: unbounded simply connected planar domains and finitely connected bounded planar domains. These generalizations are also left for a future article.

Some results for ORBM in multidimensional domains were obtained in Dupuis and Ishii (1993, 2008); Ramanan (2006); Williams (1998) under rather restrictive assumptions about smoothness of the boundary of the domain and/or the direction of reflection. The theory of non-symmetric Dirichlet forms was used to construct families of ORBMs in Kim, Kim and Yun (1998); Duarte (2012) under fairly strong assumptions. A fairly explicit formula for the stationary distribution for ORBM in a smooth planar domain was derived in Harrison, Landau and Shepp (1985). Some results on convergence of ORBMs have been recently obtained in Sarantsev (2015a,b) but the setting of those papers is considerably different from ours.

The article is organized as follows. Section 2 contains a review of some basic probabilistic and analytic facts used in the article. It also contains a theorem relating reflection vector fields on the boundary of a domain and harmonic functions inside the domain; this theorem is the fundamental analytic ingredient of our arguments. Our main results are stated in Section 3. Their proofs are given in Section 4. Our proofs are based in part on ideas developed in Burdzy and Marshall (1993).

2. Preliminaries.

2.1. *Reflected Brownian motion.* We will identify $\mathbb{C}$ and $\mathbb{R}^2$. Let $\mathcal{B}(x,r) = \{z \in \mathbb{R}^2 : |x - z| < r\}$ and $D_* = \mathcal{B}(0,1)$. Suppose that $D \subset \mathbb{C}$ is a bounded open set with smooth boundary and $\theta : \partial D \to (-\pi/2, \pi/2)$ is a Borel measurable function satisfying $\sup_{x\in\partial D} |\theta(x)| < \pi/2$. Let $\mathbf{n}(x)$ denote the unit inward normal vector at $x \in \partial D$ and let $\mathbf{t}(x) = e^{-i\pi/2}\mathbf{n}(x)$ be the unit vector tangent to ∂D at x.

Let $\mathbf{v}_\theta(x) = \mathbf{n}(x) + \tan\theta(x)\mathbf{t}(x)$, let B be standard two-dimensional Brownian motion and consider the following Skorokhod equation,

$$X_t = x_0 + B_t + \int_0^t \mathbf{v}_\theta(X_s)dL_s, \qquad \text{for } t \geq 0. \tag{2.1}$$

Here $x_0 \in \overline{D}$ and L is the local time of X on ∂D. In other words, L is a non-decreasing continuous process that does not increase when X is in D, i.e., $\int_0^\infty \mathbf{1}_D(X_t)dL_t = 0$, almost surely. If θ is C^2 then equation (2.1) has a unique pathwise solution (X, L) such that $X_t \in \overline{D}$ for all $t \geq 0$, by (Dupuis and Ishii, 1993, Cor. 5.2) (see also Dupuis and Ishii (2008)). The process X is a continuous strong Markov process on $\overline{D}_*$, and is called obliquely reflected Brownian motion in D with reflecting vector field $\mathbf{v}_\theta$. When $\theta \equiv 0$, that is, when $\mathbf{v}_\theta = \mathbf{n}$, X is called normally reflected Brownian motion in D. The goal of this paper is to construct and characterize obliquely reflected Brownian motions when θ is non-smooth and can possibly take values in $[-\pi/2, \pi/2]$, and when ∂D is also possibly non-smooth.

Consider the case when $D = D_*$ and recall that we are assuming that θ is measurable and $\|\theta\|_\infty < \pi/2$. Then one can show that (2.1) has a unique pathwise solution using the decomposition of the process in D_* into the radial and angular parts, and an argument similar to that in (Lions and Sznitman, 1984, Remark 4.2 (ii)). In both cases discussed above, the ORBM X is a strong Markov process. Since X does not visit the origin as it behaves like a Brownian motion inside the disk D_*, applying Itô's formula to $Y_t = f(X_t)$ with $f(x) = |x|$, we obtain

$$dY_t = dW_t + \frac{1}{Y_t}dt - dL_t, \tag{2.2}$$

where $W_t = \int_0^t \frac{X_s}{|X_s|} \cdot dB_s$ is a one-dimensional Brownian motion. Note that L_t increases only when $Y_t = 1$. Thus Y_t is a 2-dimensional Bessel process in $(0, 1]$ reflected at 1. It is known (see Bass and Chen (2005)) that the one-dimensional SDE (2.2) has a unique strong solution and all its weak solutions have the same distribution. It follows that the distribution of $(|X|, L)$ is independent of the reflection angle θ. Theorem 3.5 proved below implies that this property continues to hold for ORBMs in D_* with non-smooth reflection angles θ including those that could be tangential in some subset of the boundary ∂D_*.

It is known that (see Theorem 3.1(ii) below) the submartingale problem formulation of ORBM is equivalent to the one given above. Let $\mathcal{C}$ be the family of all real functions $f \in C^2(\overline{D})$ such that

$$\frac{\partial}{\partial \mathbf{n}}f(x) + \tan\theta(x)\frac{\partial}{\partial \mathbf{t}}f(x) \geq 0, \qquad x \in \partial D.$$

We will say that $\{\mathbb{P}_z : z \in \overline{D}\}$ is a solution of the submartingale problem defining an ORBM with the angle of reflection θ if $\mathbb{P}_z(X_0 = z) = 1$ for every $z \in \overline{D}$, and

$$f(X_t) - \frac{1}{2}\int_0^t \Delta f(X_s)ds, \qquad t \geq 0, \tag{2.3}$$

is a submartingale under $\mathbb{P}_z$ for every $z \in \overline{D}$ and $f \in \mathcal{C}$.

2.2. *Review of excursion theory.* We will use excursion theory of Brownian motion in our characterization of obliquely reflected Brownian motion. This section contains a brief review of the excursion theory needed in this paper. See, for example, Maisonneuve (1975) for the foundations of the theory in the abstract setting and Burdzy (1987) for the special case of excursions of Brownian motion. Although Burdzy (1987) does not discuss reflected Brownian motion, all of the results we will use from that book readily apply in the present context.

Let $\mathbb{P}_x$ denote the distribution of the process X with $X_0 = x$, defined by (2.1) or (2.3), and let $\mathbb{E}_x$ be the corresponding expectation. Let $\mathbb{P}_x^D$ denote the distribution of Brownian motion starting from $x \in D$ and killed upon exiting D.

An "exit system" for excursions of an ORBM X from ∂D is a pair (L_t^*, H^x) consisting of a positive continuous additive functional L_t^* of X and a family of "excursion laws" $\{H^x\}_{x\in\partial D}$. Let $\boldsymbol{\Delta}$ denote the "cemetery" point outside $\overline{D}$ and let $\mathcal{C}$ be the space of all functions $f : [0,\infty) \to \overline{D} \cup \{\boldsymbol{\Delta}\}$ that are continuous and take values in $\overline{D}$ on some interval $[0,\zeta)$, and are equal to $\boldsymbol{\Delta}$ on $[\zeta,\infty)$. For $x \in \partial D$, the excursion law H^x is a σ-finite (positive) measure on $\mathcal{C}$, such that the canonical process is strong Markov on (t_0,∞), for every $t_0 > 0$, with transition probabilities $\mathbb{P}_\cdot^D$. Moreover, H^x gives zero mass to paths that do not start from x. We will be concerned only with the "standard" excursion laws; see Definition 3.2 of Burdzy (1987). For every $x \in \partial D$ there exists a unique standard excursion law H^x in D, up to a multiplicative constant.

Excursions of X from ∂D will be denoted e or e_s, i.e., if $s < u$, $X_s, X_u \in \partial D$, and $X_t \notin \partial D$ for $t \in (s,u)$ then $\mathrm{e}_s = \{\mathrm{e}_s(t) = X_{t+s},\, t \in [0, u-s)\}$, $\zeta(\mathrm{e}_s) = u - s$ and $\mathrm{e}_s(t) = \boldsymbol{\Delta}$ for $t \geq \zeta$. By convention, $\mathrm{e}_t \equiv \boldsymbol{\Delta}$ if $\inf\{s > t : X_s \in \partial D\} = t$.

Let $\sigma_t = \inf\{s \geq 0 : L_s^* > t\}$ and $\mathcal{E}_u = \{\mathrm{e}_s : s < \sigma_u\}$. Let I be the set of left endpoints of all connected components of $(0,\infty)\setminus\{t \geq 0 : X_t \in \partial D\}$. The following is a special case of the exit system formula of Maisonneuve (1975). For every $x \in \overline{D}$, every bounded predictable process V_t and every universally measurable function $f : \mathcal{C} \to [0,\infty)$ that vanishes on excursions e_t identically equal to $\boldsymbol{\Delta}$, we have

$$\mathbb{E}_x\left[\sum_{t\in I} V_t \cdot f(\mathrm{e}_t)\right] = \mathbb{E}_x\left[\int_0^\infty V_{\sigma_s} H^{X(\sigma_s)}(f)ds\right] = \mathbb{E}_x\left[\int_0^\infty V_t H^{X_t}(f)dL_t^*\right]. \tag{2.4}$$

Here and elsewhere $H^x(f) = \int_{\mathcal{C}} f dH^x$. Informally speaking, (2.4) says that the right continuous version $\mathcal{E}_{t+}$ of the process of excursions is a Poisson point process on the local time scale with variable intensity $H^{\cdot}(f)$.

The normalization of the exit system is somewhat arbitrary, for example, if (L_t^*, H^x) is an exit system and $c \in (0,\infty)$ is a constant then $(cL_t^*, (1/c)H^x)$ is also an exit system. One can even make c dependent on $x \in \partial D$. Theorem 7.2 of Burdzy (1987) shows how to choose a "canonical" exit system; that theorem is stated for the usual planar Brownian motion but it is easy to check that both the statement and the proof apply to normally reflected Brownian motion (i.e., ORBM with $\theta \equiv 0$). According to that result, if D is Lipschitz then we can take L^*_t to be the continuous additive functional L^X whose Revuz measure is a constant multiple of the surface area measure dx on ∂D and H^x's to be standard excursion laws normalized so that

$$H^x(A) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{P}^D_{x+\delta \mathbf{n}(x)}(A), \tag{2.5}$$

for any event A in a σ-field generated by the process on an interval $[t_0, \infty)$, for any $t_0 > 0$. The Revuz measure of L^X is the measure $dx/(2|D|)$ on ∂D, i.e., if the initial distribution of X is the uniform probability measure μ on D, then

$$\mathbb{E}_\mu \left[\int_0^1 \mathbf{1}_A(X_s) dL^X_s \right] = \int_A \frac{dx}{2|D|}, \tag{2.6}$$

for any Borel set $A \subset \partial D$. It has been shown in Burdzy, Chen and Jones (2006) that $L^*_t = L^X_t$.

Let $K_x(\,\cdot\,)$ denote the Poisson kernel for D_*, that is, $K_x(\,\cdot\,)$ vanishes continuously on $\partial D_* \setminus \{x\}$ and is harmonic and strictly positive in D_*. We normalize K_x so that $K_x(0) = 1$ for all x. It is easy to see that the following equality holds up to a multiplicative constant,

$$\int_A K_x(y) dy = \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{E}^{D_*}_{x+\delta \mathbf{n}(x)} \left[\int_0^\infty \mathbf{1}_A(X_s) ds \right], \qquad A \subset D_*. \tag{2.7}$$

In view of (2.5), this means that $K_x(\,\cdot\,)$ is (a constant multiple of) the density of the expected occupation measure for the excursion law H^x, i.e.,

$$\int_A K_x(y) dy = H^x \left(\int_0^\infty \mathbf{1}_A(X_s) ds \right), \qquad A \subset D_*. \tag{2.8}$$

We omitted the multiplicative constant in (2.7) and (2.8) because it is equal to 1; see the proof of Theorem 3.12 (ii).

2.3. *Analytic preliminaries.* Recall that $\mathcal{B}(x,r) = \{z \in \mathbb{R}^2 : |x-z| < r\}$ and $D_* := \mathcal{B}(0,1)$. Let $\theta : \partial D_* \to [-\pi/2, \pi/2]$ be a Borel measurable function. Typically, $|dx|$ will refer to the arc length measure on ∂D_* and dz will refer to the two-dimensional Lebesgue measure on D_*. The notation $|A|$ will represent either the arc length measure of $A \subset \partial D_*$ or the two-dimensional Lebesgue measure of $A \subset D_*$; the meaning should be clear from the context. Let $\|\cdot\|_{L^1(D)}$ denote the L^1 norm for real functions on an open bounded set D with respect to two-dimensional Lebesgue measure dz on D and let $L^1(D)$ be the family of real functions in D with finite L^1 norm. We will abbreviate $\|\cdot\|_{L^1(D_*)}$ as $\|\cdot\|_1$. Similar conventions will apply to $L^\infty = L^\infty(\partial D_*)$ with respect to the measure $|dx|$ on ∂D_*. As usual, we identify functions that are equal to each other a.e. $|dx|$ on ∂D_*.

For a function f and constant c, the notation $f \not\equiv c$ will mean that f is not identically equal to c. If f is harmonic and non-negative in D_* then

$$\|f\|_1 = \int_0^1 \int_0^{2\pi} f(re^{it}) dt \; r dr = \pi f(0).$$

If the non-tangential limit of $f(z)$ at $x \in \partial D_*$ exists, we denote it by $\text{NT-lim}_{z\to x} f(z)$. If $f \in L^1(\partial D_*)$ then the harmonic extension of f to D_*, given by the Poisson integral, has nontangential limits equal to f a.e.. We will follow the usual convention of using the same letter f to denote the harmonic extension. If f is harmonic in D_*, let $\widetilde{f}$ denote the harmonic conjugate of f that vanishes at 0.

Define

$$\mathcal{T} = \{\theta \in L^\infty(\partial D_*) : \|\theta\|_\infty \le \pi/2,\ \theta \not\equiv \pi/2, \text{ and } \ \theta \not\equiv -\pi/2\},$$

$$\mathcal{B} = \{\theta : \theta \text{ is harmonic in } D_* \text{ and } |\theta(z)| < \pi/2 \text{ for all } z \in D_*\},$$

$$\mathcal{H} = \{(h,\mu_0) : h \text{ is harmonic in } D_*,\ h(z) > 0 \text{ for all } z \in D_*,\ \|h\|_1 = \pi h(0) = 1 \text{ and } \mu_0 \in \mathbb{R}\},$$

and

$$\mathcal{R} = \{\mu : \mu \text{ is harmonic in } D_* \text{ and its harmonic conjugate } \widetilde{\mu}(z) > -1 \text{ for all } z \in D_*\}.$$

The following theorem relates these spaces. See (2.23), (2.24), and Corollary 2.5 for additional formulae.

Theorem 2.1. *There are one-to-one correspondences*

$$\begin{aligned} \mathcal{T} &\leftrightarrow \mathcal{B}, & \theta(x) &\leftrightarrow \theta(z);\\ \mathcal{H} &\leftrightarrow \mathcal{R}, & (h(z),\mu_0) &\leftrightarrow \mu(z);\\ \mathcal{B} &\leftrightarrow \mathcal{H}, & \theta(z) &\leftrightarrow (h(z),\mu_0); \end{aligned}$$

given by

$$\theta(z) = \text{Re} \int_{\partial D_*} \frac{x+z}{x-z}\theta(x)\frac{|dx|}{2\pi}, \tag{2.9}$$

$$\theta(x) = \text{NT-lim}_{z\to x}\, \theta(z) \quad \textit{a.e. } |dx|, \tag{2.10}$$

$$\mu(z) = \mu_0 - \pi\widetilde{h}(z), \tag{2.11}$$

$$h(z) = (\widetilde{\mu}(z)+1)/\pi \ \textit{ and } \mu_0 = \mu(0), \tag{2.12}$$

$$h(z) = \frac{e^{\widetilde{\theta}(z)}\cos\theta(z)}{\pi\cos\theta(0)} \quad \textit{and} \quad \mu_0 = \tan\theta(0), \ \textit{and} \tag{2.13}$$

$$\theta(z) = -\arg(h(z) + i\widetilde{h}(z) - i\mu_0/\pi). \tag{2.14}$$

Moreover

$$\mu(z) = \pi h(z)\tan\theta(z) = \frac{1}{2}\lim_{r\uparrow 1}\int_{|x|=r} \text{Re}\left(\frac{x+z}{x-z}\right)h(x)\tan\theta(x)|dx| \tag{2.15}$$

and

$$\theta(z) = -\arg(h(z) - i\mu(z)/\pi). \tag{2.16}$$

PROOF. The subject of analytic and harmonic functions on the disk and their boundary values has a long history. An eminently readable reference for background material on this subject is given in the first three introductory chapters of Hoffman (1962).

Non-tangential limits give the correspondence between $\mathcal{T}$ and $\mathcal{B}$. If $\theta \in \mathcal{B}$, then θ has a non-tangential limit at almost every $x \in \partial D_*$, which we will call $\theta(x)$. The limit function $\theta(x) \in L^\infty(\partial D_*)$, and $\|\theta\|_\infty \le \pi/2$. Moreover, since $\frac{1}{2\pi}\mathrm{Re}\,\frac{x+z}{x-z}$ is the Poisson kernel on ∂D_* for $z \in D_*$, we have that

$$\theta(z) + i\widetilde{\theta}(z) = \int_{\partial D_*} \frac{x+z}{x-z}\theta(x)\frac{|dx|}{2\pi}. \tag{2.17}$$

In fact if θ is any function in L^∞ bounded by $\pi/2$ then the right-hand side (2.17) defines an analytic function on D_* whose real part is harmonic on D_*, bounded by $\pi/2$ and has non-tangential limit function $\theta(x)$, a.e. Since $\int_{\partial D_*}\theta(x)\frac{|dx|}{2\pi} = \theta(0)$, we have $\theta(x) \not\equiv \pi/2$ and $\theta(x) \not\equiv -\pi/2$ a.e. if and only if $|\theta(0)| < \pi/2$ and by the maximum principle, this occurs if and only if $|\theta(z)| < \pi/2$ for all $z \in D_*$.

If $(h, \mu_0) \in \mathcal{H}$ then μ defined by (2.11) is harmonic on D_*, with $\mu(0) = \mu_0$, and $h(z) = (\widetilde{\mu}(z) + 1)/\pi$, since $\pi h(0) = 1$ and $\widetilde{\widetilde{h}} = h(0) - h$. Since $h > 0$, we conclude that $\widetilde{\mu} > -1$ and $\mu \in \mathcal{R}$. If $\mu \in \mathcal{R}$, and if h is given by (2.12) then it is easy to verify that $(h, \mu_0) \in \mathcal{H}$. This proves the one-to-one correspondence between functions in $\mathcal{H}$ and $\mathcal{R}$.

The proof for the correspondence between $\mathcal{B}$ and $\mathcal{H}$, (2.15)-(2.16), as well as useful formulae for the corresponding harmonic conjugates are presented in the next two lemmas.

LEMMA 2.2. *There is a one-to-one correspondence between $\mathcal{B}$ and $\mathcal{H}$, $\theta \leftrightarrow (h, \mu_0)$, given by*

$$\theta + i\widetilde{\theta} = i\log(h + i\widetilde{h} - i\mu_0/\pi) - i\log\left(\left(\sqrt{1+\mu_0^2}\right)/\pi\right) \quad \textit{and} \tag{2.18}$$

$$h + i\widetilde{h} = \frac{e^{-i(\theta+i\widetilde{\theta})}}{\pi\cos\theta(0)} + i\frac{\tan\theta(0)}{\pi}, \quad \textit{and} \quad \mu_0 = \tan\theta(0). \tag{2.19}$$

PROOF. If $(h, \mu_0) \in \mathcal{H}$ then the right-hand side of (2.18) defines an analytic function $S(h, \mu_0)(z)$ on D_* with

$$\mathrm{Re}\, S(h, \mu_0)(z) = -\arg(h + i\widetilde{h} - i\mu_0/\pi) \in (-\pi/2, \pi/2)$$

and $S(h, \mu_0)(0) = -\arg(1 - i\mu_0)$, which is purely real. Thus $S(h, \mu_0) = \theta + i\widetilde{\theta}$ for some $\theta \in \mathcal{B}$. Likewise, if $\theta \in \mathcal{B}$ then the right-hand side of the first equation in (2.19) defines an analytic function, $T(\theta)(z)$, on D_* with $\mathrm{Re}\, T(\theta)(z) = e^{\widetilde{\theta}(z)}\cos\theta(z)/(\pi\cos\theta(0)) > 0$ and $\mathrm{Re}\, T(\theta)(0) = 1/\pi$. Setting $\mu_0 = \tan\theta(0)$ we conclude that if $h \equiv \mathrm{Re}\, T(\theta)$ then $(h, \mu_0) \in \mathcal{H}$. Moreover it is straightforward to verify that, given $(h, \mu_0) \in \mathcal{H}$, if θ is defined by (2.18) then

$$h = \mathrm{Re}\, T(\theta) \quad \text{and} \quad \mu_0 = \tan\theta(0).$$

Alternatively, given $\theta \in \mathcal{B}$, if (h, μ_0) is defined by (2.19) then

$$\theta = \operatorname{Re} S(h, \mu_0).$$

This proves the one-to-one correspondence in Lemma 2.2. $\square$

The equality in (2.16) of Theorem 2.1 follows immediately from (2.14) and (2.11). The first equality in (2.15) of Theorem 2.1 follows by taking real and imaginary parts in (2.19), then applying (2.11). The second equality in (2.15) follows from the Poisson integral formula on the circle of radius $r < 1$ because μ is harmonic by (2.11).

This completes the proof of Theorem 2.1. $\square$

The next lemma relates $\mu \in \mathcal{R}$ to both h and θ via a Mobius transformation. It will be used in the proof of Theorem 3.15.

LEMMA 2.3. *Suppose $(h, \mu_0) \in \mathcal{H}$, $\theta \in \mathcal{B}$, and $\mu \in \mathcal{R}$ with $(h, \mu_0) \leftrightarrow \theta \leftrightarrow \mu$. If ϕ is a one-to-one analytic map of D_* onto D_* then*

$$\theta \circ \phi \in \mathcal{B} \leftrightarrow \left(\frac{h \circ \phi}{\|h \circ \phi\|_1}, \frac{\mu(\phi(0))}{\|h \circ \phi\|_1} \right) \in \mathcal{H}. \tag{2.20}$$

PROOF. First observe that if f is harmonic then $(f + i\widetilde{f}) \circ \phi - i\widetilde{f}(\phi(0))$ is analytic with imaginary part vanishing at 0, so that

$$\widetilde{f \circ \phi} = \widetilde{f} \circ \phi - \widetilde{f}(\phi(0)). \tag{2.21}$$

Evaluating the real part of (2.19) at $z = \phi(0)$ we obtain

$$\|h \circ \phi\|_1 = \pi h(\phi(0)) = \frac{e^{\widetilde{\theta}(\phi(0))} \cos \theta(\phi(0))}{\cos \theta(0)}. \tag{2.22}$$

Set $h_1 = h \circ \phi / \|h \circ \phi\|_1 = h \circ \phi / \pi h(\phi(0))$. Then composing (2.19) with ϕ and using (2.21) and (2.11),

$$\begin{aligned} h_1 + i\widetilde{h}_1 &= \frac{h \circ \phi + i\widetilde{h} \circ \phi - i\widetilde{h}(\phi(0))}{\|h \circ \phi\|_1} \\ &= \frac{\exp(-i(\theta + i\widetilde{\theta}) \circ \phi)}{\|h \circ \phi\|_1 \pi \cos \theta(0)} + \frac{i}{\pi} \left(\frac{\tan \theta(0) - \pi \widetilde{h}(\phi(0))}{\|h \circ \phi\|_1} \right) \\ &= \frac{\exp(-i(\theta \circ \phi + i\widetilde{\theta \circ \phi}))}{\pi \cos \theta(\phi(0))} + \frac{i\mu(\phi(0))}{\pi \|h \circ \phi\|_1}. \end{aligned}$$

$\square$

By (2.19) and (2.9) the correspondence between $(h, \mu_0) \in \mathcal{H}$, $\mu \in \mathcal{R}$, and $\theta \in \mathcal{T}$ can also be written as

$$h(z) = \mathrm{Re}\left(\frac{\exp\left(-i \int_{\partial D_*} \frac{x+z}{x-z} \theta(x) \frac{|dx|}{2\pi}\right)}{\pi \cos\left(\int_{\partial D_*} \theta(x) \frac{|dx|}{2\pi}\right)} \right) \quad \text{and} \quad \mu_0 = \tan\left(\int_{\partial D_*} \theta(x) \frac{|dx|}{2\pi}\right), \tag{2.23}$$

$$\mu(z) = -\pi \mathrm{Im} \left(\frac{\exp\left(-i \int_{\partial D_*} \frac{x+z}{x-z} \theta(x) \frac{|dx|}{2\pi}\right)}{\pi \cos\left(\int_{\partial D_*} \theta(x) \frac{|dx|}{2\pi}\right)} \right). \tag{2.24}$$

We would like to have a similar formula for μ and θ in terms of h, but the situation is a little more complicated for boundary values of positive harmonic functions. A function h is positive and harmonic on D_* if and only if

$$h(z) = \int_{\partial D_*} \mathrm{Re}\left(\frac{x+z}{x-z}\right) \sigma(dx), \tag{2.25}$$

for some positive finite (regular Borel) measure σ on ∂D_*. The measures $h(rx)|dx|$ converge weakly to $\sigma(dx)$ as $r \uparrow 1$. The function h has a non-tangential limit at almost every $x \in \partial D_*$, which we will call $h(x)$, but $h(z)$ is not necessarily the Poisson integral of $h(x)$. In fact $h \to +\infty$ radially σ_s-a.e., where σ_s is the singular component of the Radon-Nikodym decomposition of σ with respect to the length measure $|dx|$ on ∂D_*. It is true, however, that a harmonic function f has non-tangential limits $f(x)$ a.e. and satisfies

$$f(z) + i\widetilde{f}(z) = \int_{\partial D_*} \frac{x+z}{x-z} f(x) \frac{|dx|}{2\pi} \tag{2.26}$$

if and only if

$$\lim_{r \uparrow 1} \int_{\partial D_*} |f(rx) - f(x)| \, |dx| = 0. \tag{2.27}$$

Given a function f defined on ∂D_* which is integrable $|dx|$, if we define $f(z)$ for $z \in D_*$ via (2.26) then f satisfies (2.27). See (Hoffman, 1962, pages 32 and 33).

If for some $p > 1$,

$$\sup_{r<1} \int_{\partial D_*} |f(rx)|^p |dx| < \infty, \tag{2.28}$$

or if

$$\sup_{r<1} \int_{\partial D_*} |(f + i\widetilde{f})(rx)| |dx| < \infty$$

then (2.27) holds. See (Hoffman, 1962, pages 33 and 51).

Example 2.4. A good example to keep in mind is

$$h(z) = \frac{1}{\pi} \mathrm{Re}\left(\frac{1+z}{1-z}\right). \tag{2.29}$$

Then $h(x) = 0$ for $x \in \partial D_* \setminus \{1\}$. So h cannot be the Poisson integral of its boundary values. Nevertheless, if $\theta \leftrightarrow (h, 0)$ then since θ is bounded, it satisfies (2.28) and hence satisfies (2.27). In fact, $\theta(x) = -\pi/2$ for $x \in \partial D_*$ with $\operatorname{Im} x > 0$ and $\theta(x) = \pi/2$ for $x \in \partial D_*$ with $\operatorname{Im} x < 0$, so that

$$\theta(z) + i\widetilde{\theta}(z) = i \log \frac{1+z}{1-z} = \int_{\partial D_*} \frac{x+z}{x-z} \theta(x)|dx|/(2\pi).$$

If h satisfies (2.27), where $(h, \mu_0) \in \mathcal{H} \leftrightarrow \theta \in \mathcal{B}$, then we can recover θ directly from the boundary values of h and μ_0. A similar result holds for μ. The following corollary will be used later to interpret $\mu(z)$ as a "rotation rate" about the point $z \in D_*$.

COROLLARY 2.5. *Suppose* $(h, \mu_0) \in \mathcal{H} \leftrightarrow \theta(z) \in \mathcal{B} \leftrightarrow \theta(x) \in \mathcal{T} \leftrightarrow \mu \in \mathcal{R}$.

(i) *If h satisfies* (2.27) *then for* $z \in D_*$

$$\theta(z) = -\arg\left(\int_{\partial D_*} \frac{x+z}{x-z} h(x) \frac{|dx|}{2\pi} - i\mu_0/\pi\right). \tag{2.30}$$

(ii) *If $h(z) \tan\theta(z)$ or $\widetilde{h}(z)$ satisfy* (2.27), *then*

$$\mu_0 = \mu(0) = \frac{1}{2} \int_{\partial D_*} h(x) \tan\theta(x)|dx|, \quad \textit{and} \tag{2.31}$$

$$\mu(z) = \frac{1}{2} \int_{\partial D_*} \operatorname{Re}\left(\frac{x+z}{x-z}\right) h(x) \tan\theta(x)|dx| \tag{2.32}$$

$$= \frac{1}{2} \int_{\partial D_*} h\Big(\frac{x+z}{1+\overline{z}x}\Big) \tan\theta\Big(\frac{x+z}{1+\overline{z}x}\Big)|dx|. \tag{2.33}$$

PROOF. (i) follows from the discussion above and (2.18).

(ii) Note that since $\mu = \mu_0 - \pi\widetilde{h}(z) = \pi h(z) \tan\theta(z)$, for $z \in D_*$, it follows that $h(z) \tan\theta(z)$ satisfies (2.27) if and only if $\widetilde{h}(z)$ satisfies (2.27). Equations (2.31) and (2.32) follow from (2.11), (2.15), and (2.26). Finally, equation (2.33) follows from (2.32) and a change of variables. □

REMARK 2.6. (i) The maps $(h, \mu_0) \to \theta$ and $\theta \to (h, \mu_0)$ are continuous under the topologies of uniform convergence on compact subsets of D_* and $(D_*, \mathbb{R})$.

(ii) For functions in $\mathcal{B}$, uniform convergence on compact subsets of D_* is equivalent to pointwise bounded convergence in D_* and is also equivalent to weak-* convergence (of the corresponding boundary value functions) in $L^\infty(\partial D_*)$, as elements of the dual space of $L^1(\partial D_*)$. But this convergence is not equivalent to pointwise bounded a.e. convergence on ∂D_*. For example, if $\theta_k(z) = -\arg(1 + z^k/2)$, then $\theta_k \leftrightarrow (h_k, 0)$, with $h_k = \operatorname{Re}(1 + z^k/2)$. The functions θ_k converge to 0, uniformly on compact subsets of D_*, pointwise boundedly on D_*, and weak-* on ∂D_*. However, θ_k does not contain a subsequence converging pointwise on any subarc in ∂D_*.

(iii) The function θ is a constant function if and only if $h \equiv 1/\pi$ and $\mu_0 = \tan\theta$. It is tempting to extend the definition of $\mathcal{T}$ to include $\theta \equiv \pi/2$ by saying $\theta \equiv \pi/2$ corresponds to $h \equiv 1/\pi$ and $\mu_0 = +\infty$. However, we would lose the continuity of the correspondence. Indeed if $(h, \mu_n), (g, \mu_n) \in \mathcal{H}$ with $\mu_n \to +\infty$ and $g \neq h$, let $\theta_{2n} \leftrightarrow (h, \mu_{2n})$ and $\theta_{2n+1} \leftrightarrow (g, \mu_{2n+1})$. Then θ_n converges to $\pi/2$ uniformly on compact subsets of D_*, but the corresponding elements of $\mathcal{H}$ do not converge.

(iv) If the pair (h, μ_0) corresponds to θ then $(h(\bar{z}), -\mu_0)$ corresponds to $-\theta(\bar{z})$. This follows from Lemma 2.2 since f is analytic if and only if $\overline{f(\bar{z})}$ is analytic. But $(h, -\mu_0)$ does not correspond to $-\theta$, unless $h \equiv 1/\pi$. Indeed, if $(h, -\mu_0)$ does correspond to $-\theta$ then

$$-(\theta + i\widetilde{\theta}) = i \log(h + i\widetilde{h} - i(-\mu_0)/\pi) - i \log \sqrt{1 + \mu_0^2}/\pi.$$

Adding this equation to (2.18) we obtain

$$0 = i \log((h + i\widetilde{h})^2 + \mu_0^2/\pi^2) - 2i \log \sqrt{1 + \mu_0^2}/\pi,$$

and thus $h + i\widetilde{h}$ is constant. Since $(h, \mu_0) \in \mathcal{H}$, we have $h \equiv h(0) = 1/\pi$.

(v) Equation (2.30) fails for the example $\theta \leftrightarrow (h, 0) \in \mathcal{H}$ where h is given by (2.29).

Example 2.7. Let $F = \phi + i\widetilde{\phi} = \sqrt{\log(1 - z^2)}$. We claim we can choose the branch of the square root so that F is analytic on D_*, with ϕ continuous on $\overline{D}_*$ and $\widetilde{\phi}$ not bounded above or below. By Theorem 2.1 and the definition of $\mathcal{R}$ there is no $(h, \mu_0) \in \mathcal{H}$ so that $\phi = \mu$, where $\mu \leftrightarrow (h, \mu_0)$. In fact there do not exist any $a, b \in \mathbb{R}$, $b \neq 0$, and $(h, \mu_0) \in \mathcal{H}$ such that $a + b\phi = \mu$. To see the claim, we set $g(z) = (\log(1 - z))/z$. Then g is analytic on a simply connected neighborhood of $\overline{D}_* \setminus \{1\}$ and non-vanishing, and hence has an analytic square root k. Then $F(z) \equiv zk(z^2)$ is analytic on a neighborhood of $\overline{D}_* \setminus \{\pm 1\}$ and satisfies $F(z)^2 = \log(1 - z^2)$. Thus ϕ and $\widetilde{\phi}$ are continuous and smooth on $\overline{D}_* \setminus \{\pm 1\}$. Since $\phi^2 - \widetilde{\phi}^2 = \log|1 - z^2| \to -\infty$ as $z \to \pm 1$, we conclude $\widetilde{\phi}^2 \to \infty$ as $z \to \pm 1$. But $2\phi\widetilde{\phi} = \arg(1 - z^2)$ is bounded, so we must have $\phi \to 0$ as $z \to \pm 1$. Thus ϕ is continuous on $\overline{D}_*$, and $\widetilde{\phi}$ is unbounded. Since F is odd, $\widetilde{\phi}$ is neither bounded above nor below.

Example 2.8. Consider the harmonic function $\phi(z) = \mathrm{Re}\, z$ in D_* with boundary values $\phi(e^{it}) = \cos t$, $0 \leq t < 2\pi$. If $a, b \in \mathbb{R}$, with $b \neq 0$, set $\mu = a + b\phi = a + b\mathrm{Re}\, z$. Then $\widetilde{\mu} = b\mathrm{Im}\, z > -1$ for all $z \in D_*$ if and only if $|b| \leq 1$. By the equivalence of $\mathcal{R}$ and $\mathcal{H}$ given in Theorem 2.1, $\mu = a + b\phi$ corresponds to some $(h, \mu_0) \in \mathcal{H}$ if and only if $|b| \leq 1$.

If ϕ is harmonic on D_* and if $\widetilde{\phi}$ is bounded, then for $a, b \in \mathbb{R}$ with $b \neq 0$, the function $\mu = a + b\phi$ has harmonic conjugate $b\widetilde{\phi}$. So for sufficiently small b, we have $\widetilde{\mu} > -1$ which implies $\mu \in \mathcal{R}$ and $a + b\phi \leftrightarrow (h, \mu_0) \in \mathcal{H}$ for some (h, μ_0). Since $\widetilde{\mu}(0) = 0$, we have that $\inf \widetilde{\phi} < 0 < \sup \widetilde{\phi}$ so that for $|b|$ sufficiently large $\mu = a + b\phi$ fails to be in $\mathcal{R}$. So in some sense, membership in $\mathcal{R}$ depends on the "oscillation" of the harmonic function on D_*, but not its mean. The next proposition gives a more precise version. Its proof is elementary, but it will be useful for understanding our (later) description of rotation rates and stationary distributions for ORBMs.

PROPOSITION 2.9. *Suppose ϕ is (real-valued and) harmonic in D_*. Set*

$$K_- = \inf_{z\in D_*} \widetilde{\phi}(z) \quad \text{and} \quad K_+ = \sup_{z\in D_*} \widetilde{\phi}(z)$$

If $a, b \in \mathbb{R}$ with $-1/|K_+| \le b \le 1/|K_-|$, then there is a unique $(h, \mu_0) \in \mathcal{H}$ such that

$$a + b\phi(z) = \mu(z), \tag{2.34}$$

where μ and (h, μ_0) are related as in Theorem 2.1. Conversely, if $b < -1/|K_+|$ or $b > 1/|K_-|$ then there do not exist any $a \in \mathbb{R}$ and $(h, \mu_0) \in \mathcal{H}$ such that (2.34) holds.

In the statement of Proposition 2.9 we allow the possibility that K_+ is infinite, in which case we interpret $1/|K_+|$ as equal to zero. A similar statement holds for $|K_-|$.

PROOF. Note that $K_- \le 0 \le K_+$ since $\widetilde{\phi}(0) = 0$. If $b \in \mathbb{R}$ and if $-1/|K_+| \le b \le 1/|K_-|$, set $\mu = a + b\phi$. Then $\widetilde{\mu}(z) = b\widetilde{\phi}(z) \ge -1$. Since $\widetilde{\mu}(0) = 0$, the maximum principle implies that $\widetilde{\mu}(z) > -1$ for all $z \in D_*$, so that $\mu \in \mathcal{R}$. The corresponding $(h, \mu_0) \in \mathcal{H}$ is given by (2.12) of Theorem 2.1.

Conversely if $(h, \mu_0) \in \mathcal{H}$ corresponds to $\mu = a + b\phi \in \mathcal{R}$ as in Theorem 2.1, then $\widetilde{\mu}(z) = b\widetilde{\phi}(z) > -1$. But this implies $b \ge -1/\sup \widetilde{\phi}(z)$ and $b \le 1/|\inf \widetilde{\phi}(z)|$. □

If a real-valued function is slightly better than continuous, then its harmonic conjugate is continuous and hence bounded. For a function $f : \partial D_* \to \mathbb{R}$, we define the modulus of continuity of f by $\omega_f(a) = \sup_{|s-t|<a} |f(e^{is}) - f(e^{it})|$. We say that f is Dini continuous if $\int_0^b (\omega_f(a)/a)da < \infty$ for some $b > 0$. If f is Dini continuous then $\widetilde{f}$ is continuous and therefore bounded. See (Garnett, 2007, Thm III.1.3).

THEOREM 2.10. *Suppose that $\theta \in \mathcal{T}$, $(h, \mu_0) \in \mathcal{H}$, and $\mu \in \mathcal{R}$ correspond to each other as in Theorem 2.1. See also (2.23) and (2.24).*

(i) *If θ is Dini continuous on ∂D_*, then h and μ extend to be continuous on $\overline{D}_*$. If μ is Dini continuous on ∂D_*, then h is continuous on $\overline{D}_*$ and θ is continuous on $\overline{D}_* \setminus Z$, where $Z = \{x \in \partial D_* : h(x) = \mu(x) = 0\}$. Similarly, if h is Dini continuous on ∂D_*, then μ is continuous on $\overline{D}_*$, and θ is continuous on $\overline{D}_* \setminus Z$. In each of these cases, h and $\widetilde{h}$ satisfy (2.27), so that the conclusions of Corollary 2.5 hold.*

(ii) *Suppose that ω is an increasing continuous concave function on $[0, \pi/2]$ such that $\omega(0) = 0$, $\omega(\pi/2) = \pi/4$, and $\int_0^{\pi/2} \frac{\omega(a)}{a} da = \infty$. Then there exists $\theta \in \mathcal{T}$ such that its modulus of continuity $\omega_\theta(a) = \omega(a)$ for $a \in [0, \pi/2]$ and both h and μ are unbounded.*

PROOF. (i) By (Garnett, 2007, Thm. III.1.3), if θ is Dini continuous then the harmonic conjugate $\widetilde{\theta}$ is continuous on $\overline{D}_*$. Hence, $F(z) = \exp(\widetilde{\theta}(z) - i\theta(z))$ is continuous and so is $h + i\widetilde{h}$ by (2.19). Hence h and $\mu = \mu_0 - \pi\widetilde{h}$ are continuous. The remaining statements in (i) follow from (2.11), (2.12), and (2.18) and (Garnett, 2007, Cor. III.1.4). In each of the cases in (i), h and $\widetilde{h}$ are continuous on $\overline{D}_*$ and hence satisfy (2.27).

(ii) We give here an example based on (Garnett, 2007, page 101). Suppose that ω is increasing and concave on $[0, \pi/2]$ with $\omega(0) = 0$, $\omega(\pi/2) = \pi/4$, and

$$\int_0^{\pi/2} (\omega(t)/t)dt = \infty. \tag{2.35}$$

Set

$$\alpha(t) = \begin{cases} \omega(t) & \text{if } 0 \le t \le \pi/2, \\ \omega(\pi - t) & \text{if } \pi/2 \le t \le \pi, \\ 0 & \text{if } -\pi < t < 0. \end{cases}$$

For $0 \le x < y \le \pi$, write $x = ty,\ \ 0 < t < 1$, and so $y - x = (1-t)y$. Since ω is concave and $\alpha(0) = \omega(0) = 0$,

$$t\alpha(y) \le \alpha(x) \qquad \text{and} \qquad (1-t)\alpha(y) \le \alpha(y-x).$$

Adding these inequalities we obtain $\alpha(y) - \alpha(x) \le \alpha(y-x)$. Since $\alpha(\pi) = 0$, replacing $\alpha(t)$ by $\alpha(\pi - t)$ in the above argument, we also have that $\alpha(x) - \alpha(y) \le \alpha(y-x)$. If $x < 0 < y < \pi$ with $|x - y| < \pi/2$, then

$$\alpha(y) - \alpha(x) = \alpha(y) \le \alpha(y + |x|) = \alpha(y - x).$$

Set $\theta(e^{it}) = -\alpha(t)$. Then $\theta \in \mathcal{T}$, because $|\alpha| \le \pi/4$, and $\omega_\theta(a) = \omega_\alpha(a) = \omega(a)$ for $0 \le a \le \pi/2$.

Let $b(r) = \cos^{-1}(\frac{1+r}{2})$. Then for $r \in (0,1)$,

$$\begin{aligned} \widetilde{\theta}(r) &= -\frac{1}{2\pi}\int_0^\pi \operatorname{Im}\left(\frac{e^{it}+r}{e^{it}-r}\right)\alpha(t)dt \\ &\ge \frac{1}{2\pi}\int_{b(r)}^\pi \frac{2r\sin t}{|e^{it}-r|^2}\alpha(t)dt. \end{aligned}$$

Since $|e^{it} - 1| \ge |e^{it} - r|$ when $\cos t \le (1+r)/2$, we have that

$$\widetilde{\theta}(r) \ge -\frac{r}{2\pi}\int_{b(r)}^\pi \operatorname{Im}\left(\frac{e^{it}+1}{e^{it}-1}\right)\alpha(t)dt = \frac{r}{2\pi}\int_{b(r)}^\pi \frac{\alpha(t)}{\tan t/2}dt,$$

which increases to $+\infty$ as $r \to 1$. So $\widetilde{\theta}(r)$ is not bounded above. Because θ is continuous on ∂D_* with $\theta(1) = 0,\ \ \theta(z)$ extends to be continuous on $\overline{D}_*$ and $\cos\theta(r) \to 1$ as $r \to 1$, so by (2.13) h is also unbounded. □

Theorem 2.10 (ii) implies that if $\theta \in \mathcal{T}$ is not Dini continuous on ∂D_*, then h and μ may not be extended continuously to $\overline{D}_*$. The next proposition examines the situation when θ is as large as possible on an interval of ∂D_*.

Proposition 2.11. *Suppose I is an open arc in ∂D_*, and suppose $\theta \in \mathcal{T} \leftrightarrow (h, \mu_0) \in \mathcal{H}$.*

(i) *If $\theta(x) = \pi/2$ a.e. on I, then $f = h + i\widetilde{h} - i\mu_0/\pi$ extends to be analytic in a neighborhood of $D_* \cup I$ with $h = 0$ on I. The same conclusion holds if $\theta(x) = -\pi/2$ a.e. on I.*

(ii) *If h extends to be continuous on $D_* \cup I$ with $h = 0$ on I, then $f = h + i\widetilde{h} - i\mu_0/\pi$ extends to be analytic in a neighborhood of $D_* \cup I$ with at most one zero $e^{it_0} \in I$. If $f \neq 0$ on I then $\theta \equiv \pi/2$ or $\theta \equiv -\pi/2$ on I. If $f(e^{it_0}) = 0$ for some $e^{it_0} \in I$, then $\theta(e^{it}) = -\pi/2$ for $e^{it} \in I$ with $t < t_0$ and $\theta(e^{it}) = \pi/2$ for $e^{it} \in I$ with $t > t_0$.*

PROOF. (i) Suppose $\theta(x) = \pi/2$ a.e. on I. For $z \in D_*$ set $F(z) = \theta(z) - \pi/2 + i\widetilde{\theta}(z)$. Then by (2.17)

$$F(z) = \int_{\partial D_*} \frac{x+z}{x-z}(\theta(x) - \pi/2)\frac{|dx|}{2\pi} = \int_{\partial D_* \setminus I} \frac{x+z}{x-z}(\theta(x) - \pi/2)\frac{|dx|}{2\pi}. \tag{2.36}$$

The right-hand side of (2.36) defines an analytic function on $\mathbb{C} \setminus (\partial D_* \setminus I)$. By (2.19), $f \equiv h + i\widetilde{h} - i\mu_0/\pi$ extends to be analytic in a neighborhood of $D_* \cup I$. Also by (2.36)

$$\operatorname{Re} F(z) = \theta - \pi/2 = \int_{\partial D_* \setminus I} \frac{1 - |z|^2}{|x-z|^2}(\theta(x) - \pi/2)\frac{|dx|}{2\pi}.$$

If $y \in I$, then $\frac{1-|z|^2}{|x-z|^2} \to 0$ uniformly in $x \in \partial D_* \setminus I$ as $z \to y$. Thus $\operatorname{Re} F(z) = \theta(z) - \pi/2 \to 0$ as $z \to y \in I$. Taking real part of (2.19),

$$h(z) = \frac{e^{\widetilde{\theta}(z)} \cos\theta(z)}{\pi \cos\theta(0)},$$

so by the continuity of θ and $\widetilde{\theta}$ on $D_* \cup I$, we have $h \to 0$ as $z \to y \in I$.

To prove (ii), suppose that h extends to be continuous on $D_* \cup I$ with $h = 0$ on I. By the Schwarz reflection principle $f = h + i\widetilde{h} - i\mu_0/\pi$ extends analytically across I. By the Cauchy-Riemann equations,

$$\frac{\partial}{\partial t}\operatorname{Im} f(e^{it}) = \frac{\partial}{\partial r}\operatorname{Re} f(re^{it})|_{r=1} = \frac{\partial h}{\partial r} \leq 0$$

on I since $h = 0$ on I and $h > 0$ on D_*. Since $\operatorname{Re} f = 0$ on I, $\operatorname{Im} f$ cannot be constant on any subarc of I and thus f is a one-to-one map of the arc I onto a subarc of the imaginary axis, and (*ii*) follows from (2.30). □

3. Main results. This section contains only statements of the main results of this paper. The proofs will be given in Section 4. First, in Section 3.1, we establish results when the domain D is smooth and the angle of reflection θ is C^2 and non-tangential everywhere, that is, θ lies in a closed subinterval of $(-\pi/2, \pi/2)$. Theorem 3.1 summarizes results on existence and uniqueness of ORBMs, and Theorem 3.2 considers ORBMs on the disk D_* and establishes the probabilistic interpretation of the quantity $(h(z), \mu_0)$ corresponding to $\theta \in \mathfrak{T}$, as specified in Theorem 2.1. ORBMs in D_* with general reflection angles $\theta \in \mathfrak{T}$ are constructed in Section 3.2. The focus of Section 3.3 (in particular, see Theorem 3.12) is the case when the reflection vector field is tangential at every point, which leads to a process referred to as excursion reflected Brownian motion (ERBM). Lastly, in Section 3.4 (specifically, Theorems 3.15–3.18 therein) we construct ORBMs in simply connected domains using conformal mappings and then show, in the case of simply connected bounded Jordan domains, that they can also be obtained as suitable limits of ORBMs in C^2 domains.

3.1. *Smooth D and C^2-smooth non-tangential θ.* We start with a theorem on existence and uniqueness of ORBM in the simplest case, when the domain is smooth and the angle of reflection is smooth and takes values in a closed subinterval of $(-\pi/2, \pi/2)$. The result is essentially known.

THEOREM 3.1. *Assume that $D \subset \mathbb{C}$ is a bounded open set with C^2 boundary, and a function $\theta : \partial D \to (-\pi/2, \pi/2)$ is C^2.*

(i) ((Harrison, Landau and Shepp, 1985, Thm. 2.6)) *The submartingale problem* (2.3) *has a unique solution which defines a strong Markov process.*
(ii) *The strong Markov process defined by the Skorokhod equation* (2.1) *is continuous and has the same distribution as the process defined by the submartingale problem* (2.3).
(iii) (Kim, Kim and Yun (1998)) *The ORBM obtained in (i) and (ii) can also be constructed by using the non-symmetric Dirichlet form approach.*

It follows from the results in Harrison, Landau and Shepp (1985) that if θ is C^1 then the ORBM X in the unit disc D_* has a unique stationary distribution with the density h given by (2.23). The stationary distribution was characterized in Harrison, Landau and Shepp (1985) in terms of a partial differential equation in D_* with appropriate boundary conditions. In Theorem 3.2 (ii), we will show a partial converse, namely, that the stationary distribution characterizes an ORBM up to a real number that represents the "rotation rate" of X about 0.

Under the assumptions of Theorem 3.1, the ORBM X is continuous, a.s.. Consider a fixed $z \in D_*$. Since $X_t \neq z$ for all $t > 0$, a.s. (even if $X_0 = z$), we can uniquely define the function $t \to \arg(X_t - z)$ by choosing its continuous version and making an arbitrary convention that $\arg(X_1 - z) \in [0, 2\pi)$.

Since h is the density of the stationary measure of X and θ is the reflection angle, (2.31) suggests that μ_0 represents one half of the speed of rotation of X about 0. Hence, one might hope that $\lim_{t\to\infty} \arg X_t/t$ is equal to a constant multiple of μ_0, a.s. Unfortunately, this simple interpretation of μ_0 is false because $\arg X_t$ behaves like a Cauchy process (see Spitzer (1958); Bertoin and Werner (1994)) and, therefore, the law of large numbers does not hold for $\arg X_t$. We will identify μ_0 with the speed of rotation using two other representations in Theorem 3.2 (ii)-(iii). We need the following definitions to state the representations. First of all, recall that a random variable has the Cauchy distribution if its density is $1/(\pi(1 + x^2))$ for $x \in \mathbb{R}$. Next we will define a new measure of winding speed which does not include large windings if they occur during a single excursion from the boundary. Recall definitions related to excursions from Section 2.2. We will say that e_s belongs to the family $\mathcal{E}^L_t$ of excursions with "large winding number" if $s + \zeta(\mathrm{e}_s) \le t$ and $|\arg X_s - \arg X_{s+\zeta(\mathrm{e}_s)-}| > 2\pi$, where X_{u-} denotes the left-hand limit. For $z \in D_*$, let

$$\arg^* X_t = \arg X_t - \sum_{s:\, \mathrm{e}_s \in \mathcal{E}^L_t} \left(\arg X_{s+\zeta(\mathrm{e}_s)-} - \arg X_s\right), \tag{3.1}$$

$$\arg^*(X_t - z) = \arg(X_t - z) - \sum_{s:\, \mathrm{e}_s \in \mathcal{E}^L_t} \left(\arg(X_{s+\zeta(\mathrm{e}_s)-} - z) - \arg(X_s - z)\right). \tag{3.2}$$

THEOREM 3.2. *In parts (i)-(iii), we assume that a C^2 function $\theta : \partial D_* \to (-\pi/2, \pi/2)$ is given.*

(i) ((Harrison, Landau and Shepp, 1985, Thm. 2.18)) *The density of the stationary measure for* X *defined in* (2.1) *is a positive harmonic function* h *in* D_* *given by* (2.23) *(see also* (2.19)*).*

(ii) *With probability 1,* X *is continuous and, therefore,* $\arg X_t$ *is well defined for* $t > 0$*. Let* $\mu_0 \in \mathbb{R}$ *be given by* (2.23)*. For every* $z \in \overline{D}_*$*, the distributions of* $\frac{1}{t}\arg X_t - \mu_0$ *under* $\mathbb{P}_z$ *converge to the Cauchy distribution when* $t \to \infty$*.*

(iii) *For every* $y \in \overline{D}_*$*,*

$$\lim_{t\to\infty} \frac{1}{t}\arg^* X_t = \mu_0, \quad \mathbb{P}_y\text{-a.s.} \tag{3.3}$$

The formula holds more generally. For any $y, z \in D_*$*,*

$$\lim_{t\to\infty} \frac{1}{t}\arg^*(X_t - z) = \mu(z), \quad \mathbb{P}_y\text{-a.s.,} \tag{3.4}$$

where $\mu(z)$ *is given by* (2.24)*.*

(iv) *Conversely, suppose we are given any* $\mu_0 \in \mathbb{R}$ *and a harmonic function* h *in* D_* *that is* C^2 *in* $\overline{D}_*$*, positive on* $\overline{D}_*$*, and satisfies* $h(0) = 1/\pi$*. Let* $\theta \leftrightarrow (h, \mu_0)$*. Then for every* $x_0 \in \overline{D}_*$*, there exists a unique in distribution process* X *satisfying* (2.1) *with this* θ*. Its stationary distribution has density* h *and* (3.3) *holds.*

REMARK 3.3. (i) We could have defined the family $\mathcal{E}_t^L$ of excursions e_s with "large winding number" as those satisfying $s + \zeta(\mathrm{e}_s) \le t$ and $|\arg X_s - \arg X_{s+\zeta(\mathrm{e}_s)-}| > a$, where $a > 0$ is not necessarily 2π. It turns out that (3.3) holds for any $a > 0$. The limit in (3.3) holds for any value of a because the only thing that matters in (3.1) is that the large jumps of the Cauchy-like process $\arg X$ are removed. The "remaining part" of this process satisfies the law of large numbers and has mean $\mu_0 t$, no matter how large the threshold for the "large jumps" is. We have chosen $a = 2\pi$ because this value has a natural geometric interpretation and is invariant, in a sense, under conformal mappings.

(ii) We will prove (3.4) using (3.23) and a purely analytic argument. Formula (3.4) has the same heuristic meaning as (2.31) as a rotation rate, except that it represents the sum (integral) of infinitesimally small increments of the angle around z, not 0.

(iii) In view of Theorem 2.1, if the rotation rate $\mu(z)$ is known for all $z \in D_*$, it completely determines θ and h. Moreover, due to the harmonic character of $\mu(z)$, if this function is known in an arbitrarily small non-empty open subset of D_*, this also determines θ and h.

(iv) Theorem 2.1 and the definition of the function space $\mathcal{R}$ show which harmonic functions $\mu(z)$ represent rotation rates for an ORBM. See also Proposition 2.9. Roughly speaking, $\mu(z)$ represents rotation rates for an ORBM if its oscillation over $\overline{D}_*$ is not too large. There is no restriction, however, on the average value of $\mu(z)$. If $\mu(z)$ and $\mu_1(z)$ represent the rotation rates for two ORBM's, and $\mu(z) = c + \mu_1(z)$ for some constant c and all z then $\widetilde{\mu} = \widetilde{\mu}_1$. By (2.12) of Theorem 2.1, the corresponding stationary densities are the same for both ORBM's.

(v) Parts (ii) and (iii) of Theorem 3.2 are similar in spirit to (Le Gall and Yor, 1986, Thm. 7.1) although that paper is concerned with Brownian motion with drift, not reflection.

3.2. *ORBMs on D_* with general reflection angles θ.* Suppose $\theta \in \mathcal{T}$. Then $\theta \not\equiv \pi/2$ and $\theta \not\equiv -\pi/2$, although θ could be tangential on a strict subset of the boundary ∂D_*. In Theorem 3.5 we show that ORBMs on the disk D_* associated with θ can be obtained as limits of ORBMs on D_* with C^2 angles of reflection, which are well defined by Theorem 3.1. Then in Theorem 3.8 we establish a conformal invariance property for such ORBMs. If there do exist points on the boundary at which θ is tangential, the associated ORBM will not in general be continuous, and thus one has to carefully define the topology in which the above limit procedure can be carried out.

We start by introducing some relevant notation to define this topology. Let

$$N_\theta^+ = \{x \in \partial D_* : \theta(x) = \pi/2\}, \qquad N_\theta^- = \{x \in \partial D_* : \theta(x) = -\pi/2\}. \tag{3.5}$$

Since we identify functions in $\mathcal{T}$ that are equal to each other a.e.,

$$|N_\theta^+| < 2\pi \qquad \text{and} \qquad |N_\theta^-| < 2\pi. \tag{3.6}$$

We will say that $x \in \operatorname{Int} N_\theta^+$ if $\theta \equiv \pi/2$ a.e. in some neighborhood of x. The definition of $\operatorname{Int} N_\theta^-$ is analogous. For $x = e^{i\alpha} \in \operatorname{Int} N_\theta^+$, let α^+ be the largest real number such that $\{e^{it} : t \in [\alpha, \alpha^+)\} \subset \operatorname{Int} N_\theta^+$, and let $\beta^+(x) = e^{i\alpha^+}$. Similarly, for $x = e^{i\alpha} \in \operatorname{Int} N_\theta^-$, let α^- be the smallest real number such that $\{e^{it} : t \in (\alpha^-, \alpha]\} \subset \operatorname{Int} N_\theta^-$, and let $\beta^-(x) = e^{i\alpha^-}$.

We recall below the definition of the M_1 topology introduced by Skorokhod in Skorokhod (1956). We will use the M_1 topology rather than the more popular J_1 topology because we will be concerned with convergence of continuous processes to (possibly) discontinuous processes. In the J_1 topology, a sequence of continuous processes cannot converge to a discontinuous process. We will also define an $M_1^{\mathcal{T}}$ topology, appropriate for our setting.

Definition 3.4. (i) Suppose that $0 < T < \infty$ and $x : [0, T] \to \mathbb{R}^n$ is a càdlàg function. The graph Γ_x is the set consisting of all pairs (a, t) such that $0 \le t \le T$ and $a \in [x(t-), x(t)]$ (here $[x(t-), x(t)]$ is the line segment between the left-hand limit $x(t-)$ and $x(t)$ in $\mathbb{R}^n$). A pair of functions $\{(y(s), t(s)), s \in [0, 1]\}$ is a parametric representation of Γ_x if y is continuous, t is continuous and non-decreasing, and $(v, u) \in \Gamma_x$ if and only if $(v, u) = (y(s), t(s))$ for some $s \in [0, 1]$. We say that x_n *converge to x in M_1 topology* if there exist parametric representations $\{(y(s), t(s)), s \in [0, 1]\}$ of Γ_x and $\{(y_n(s), t_n(s)), s \in [0, 1]\}$ of Γ_{x_n} such that

$$\lim_{n\to\infty} \sup_{s\in[0,1]} |(y_n(s), t_n(s)) - (y(s), t(s))| = 0. \tag{3.7}$$

(ii) If $x : [0, \infty) \to \mathbb{R}^n$ then we say that $x_n(t)$ *converge to $x(t)$ in M_1 topology* if they converge to x on $[0, T]$ in M_1 topology for every $0 < T < \infty$. This is equivalent to the following statement. There exist parametric representations $\{(y(s), t(s)), s \in [0, \infty)\}$ of Γ_x and $\{(y_n(s), t_n(s)), s \in [0, \infty)\}$ of Γ_{x_n} such that for every $T \in (0, \infty)$,

$$\lim_{n\to\infty} \sup_{s\in[0,T]} |(y_n(s), t_n(s)) - (y(s), t(s))| = 0. \tag{3.8}$$

(iii) Consider $\theta \in \mathcal{T}$. We will say that $x : [0, \infty) \to \overline{D}_*$ *belongs to $\mathcal{A}_\theta$* if it is càdlàg and satisfies the following conditions. For all $t \ge 0$, $x_{t-} \neq x_t$ if and only if $x_{t-} \in \operatorname{Int} N_\theta^+ \cup \operatorname{Int} N_\theta^-$. Moreover, if $x_{t-} \in \operatorname{Int} N_\theta^+$ then $x_t = \beta^+(x_{t-})$. If $x_{t-} \in \operatorname{Int} N_\theta^-$ then $x_t = \beta^-(x_{t-})$. Let $\mathcal{A}_{\mathcal{T}} = \bigcup_{\theta\in\mathcal{T}} \mathcal{A}_\theta$.

(iv) Assume that $\theta \in \mathcal{T}$ and $x \in \mathcal{A}_\theta$. If $x_{t-} = e^{i\alpha} \in \operatorname{Int} N_\theta^+$ and $x_t = \beta^+(x_{t-}) = e^{i\alpha^+}$, then we let $[x_{t-}, x_t]_\theta = \{e^{it} : t \in [\alpha, \alpha^+]\}$ be the arc on ∂D_* between x_{t-} and x_t. Thus $\theta(e^{is}) = \pi/2$ for a.e. $e^{is} \in [x_{t-}, x_t]_\theta$. Similarly, if $x_{t-} = e^{i\alpha} \in \operatorname{Int} N_\theta^-$ and $x_t = \beta^-(x_{t-}) = e^{i\alpha^-}$, then we let $[x_{t-}, x_t]_\theta = \{e^{it} : t \in [\alpha^-, \alpha]\}$.

We define the graph Γ_x^θ as the set of all pairs (a,t) such that $a = x_t$ if x is continuous at t and $a \in [x_{t-}, x_t]_\theta$ if $x_{t-} \neq x_t$. A pair of functions $\{(y(s), t(s)), s \in [0,\infty)\}$ is a parametric representation of Γ_x^θ if y is continuous, t is continuous and non-decreasing, and $(v,u) \in \Gamma_x^\theta$ if and only if $(v,u) = (y(s), t(s))$ for some $s \in [0,\infty)$. Suppose that $x_n \in \mathcal{A}_{\theta_n}$ for some $\theta_n \in \mathcal{T}$, $n \geq 1$, and $x \in \mathcal{A}_\theta$ for some $\theta \in \mathcal{T}$. We say that x_n converge to x in $M_1^{\mathcal{T}}$ topology if there exist parametric representations $\{(y(s), t(s)), s \in [0,\infty)\}$ of Γ_x^θ and $\{(y_n(s), t_n(s)), s \in [0,\infty)\}$ of $\Gamma_{x_n}^{\theta_n}$ such that for every $T \in (0,\infty)$,

$$\lim_{n\to\infty} \sup_{s\in[0,T]} |(y_n(s), t_n(s)) - (y(s), t(s))| = 0. \tag{3.9}$$

Some càdlàg functions x (for example, continuous functions) belong to more than one family $\mathcal{A}_\theta$. We leave it to the reader to check that the definitions in (iv) are not affected by the choice of $\mathcal{A}_\theta$.

We will extend the definition of $t \to \arg X_t$ to (some) processes that are not continuous. Although it is impossible to define a continuous version of $t \to \arg X_t$ for a process X that is discontinuous, we will define a functional $\{X_t, t \geq 0\} \to \{\mathbf{arg}\, X_t, t \geq 0\}$ in a way that reflects the structure of jumps in a natural way, leading to heuristically appealing results. The functional **arg** will be defined relative to θ but the dependence will be suppressed in the notation. Consider a function $x \in \mathcal{A}_\theta$ such that $x_t \neq 0$ for all $t \geq 0$. Consider any parametric representation $\{(y(s), t(s)), s \in [0,\infty)\}$ of Γ_x^θ and let $s \to \arg y(s)$ be the continuous version of $\arg y$ with $\arg y(0) \in [0, 2\pi)$. We let $\mathbf{arg}\, x_u = \arg y(s)$ where $s = \sup\{r : t(r) = u\}$. It is elementary to check that this definition of $\mathbf{arg}\, x_u$ does not depend on the choice of parametric representation $\{(y(s), t(s)), s \in [0,\infty)\}$ of Γ_x^θ.

Recall the definitions (3.1)-(3.2) and notation introduced in the paragraph preceding them. We define $\mathbf{arg}^*$ in an analogous way. For $z \in D_*$, let

$$\mathbf{arg}^* X_t = \mathbf{arg}\, X_t - \sum_{s:\, \mathrm{e}_s \in \mathcal{E}_t^L} \left(\mathbf{arg}\, X_{s+\zeta(\mathrm{e}_s)-} - \mathbf{arg}\, X_s \right),$$

$$\mathbf{arg}^*(X_t - z) = \mathbf{arg}(X_t - z) - \sum_{s:\, \mathrm{e}_s \in \mathcal{E}_t^L} \left(\mathbf{arg}(X_{s+\zeta(\mathrm{e}_s)-} - z) - \mathbf{arg}(X_s - z) \right).$$

THEOREM 3.5. *Consider $\theta \in \mathcal{T}$. There exists a sequence of C^2 functions $\theta_k : \partial D_* \to (-\pi/2, \pi/2)$ which converges to θ in weak-* topology as elements of the dual space of $L^1(\partial D_*)$, that is,*

$$\lim_{k\to\infty} \int_{\partial D_*} f(x)\theta_k(x)|dx| = \int_{\partial D_*} f(x)\theta(x)|dx| \quad \textit{for every } f \in L^1(\partial D_*).$$

Fix such a sequence $\{\theta_k\}$ and let X^k be defined by the following SDE analogous to (2.1),

$$X_t^k = z_k + B_t + \int_0^t \mathbf{v}_{\theta_k}(X_s^k) dL_s^k \qquad \textit{for } t \geq 0. \tag{3.10}$$

Assume that $z_k \to z_0 \in D_*$ as $k \to \infty$, $z_0 \neq 0$, and recall (3.6).

(i) ((Burdzy and Marshall, 1993, Thm. 1.1)) *X^k's converge weakly in $M_1^{\mathcal{T}}$ topology to a conservative Markov process X on $\overline{D}_*$ such that $X_0 = z_0$, a.s. Moreover, there is a càdlàg version of X and for this version, $X \in \mathcal{A}_\theta$, a.s. The process $\{X_t; t \in [0, \sigma_{\partial D_*})\}$, where $\sigma_{\partial D_*} := \inf\{t > 0 : X_t \in \partial D_*\}$, is Brownian motion killed upon leaving D_*.*

(ii) *X^k's converge to X in the sense of finite dimensional distributions.*

(iii) *The Markov process X has a stationary measure whose density h is given by* (2.23)*.*

(iv) *The functional $\{x_s, s \in [0,\infty)\} \to \{\mathbf{arg}\, x_s, s \in [0,\infty)\}$ is a continuous mapping from the set $\mathcal{A}_{\mathcal{T}}$ equipped with $M_1^{\mathcal{T}}$ topology to the set of càdlàg functions equipped with the M_1 topology. For every $t \geq 0$, the distributions of $\mathbf{arg}\, X_t^k$ converge to the distribution of $\mathbf{arg}\, X_t$.*

(v) *Let μ_0 be as in* (2.23)*. Then for every $z \in \overline{D}_*$, the distributions of $\frac{1}{t}\mathbf{arg}\, X_t - \mu_0$ under $\mathbb{P}_z$ converge to the Cauchy distribution when $t \to \infty$.*

(vi) *For every $y \in \overline{D}_*$, $\mathbb{P}_y$-a.s.,*

$$\lim_{t\to\infty} \frac{1}{t} \mathbf{arg}^* X_t = \mu_0. \tag{3.11}$$

Moreover, for any $y, z \in D_*$,

$$\lim_{t\to\infty} \frac{1}{t} \mathbf{arg}^*(X_t - z) = \mu(z), \qquad \mathbb{P}_y\text{-a.s.}, \tag{3.12}$$

where $\mu(z)$ is the harmonic function defined by (2.24)*.*

(vii) *Assume that $\theta \in \mathcal{T} \leftrightarrow (h, \mu_0) \in \mathcal{H}$. Then for every $x \in \partial D_*$, $x \in \Gamma_X^\theta$ with probability 1 if and only if*

$$\int_0^1 e^{-\widetilde{\theta}(rx)} \cos\theta(rx) \frac{dr}{1-r} = \int_0^1 \frac{h(rx)/(\pi\cos\theta(0))}{h(rx)^2 + (\widetilde{h}(rx) - \mu_0/\pi)^2} \frac{dr}{1-r} < \infty. \tag{3.13}$$

(viii) *Suppose that $\theta, \bar{\theta}_k \in \mathcal{T}$ and $\bar{\theta}_k$ converge to θ in weak-* topology. Let $\bar{X}^k$'s have their distributions determined by $\bar{\theta}_k$'s in the same way as X's distribution is determined by θ. Assume that $\bar{X}_0^k = z_k$, $X_0 = z_0$ and $z_k \to z_0$ as $k \to \infty$. Then $\bar{X}^k$ converge weakly to X in $M_1^{\mathcal{T}}$ topology.*

We will call the process X obtained in Theorem 3.5 ORBM with reflection angle θ.

Remark 3.6. (i) Note that the distribution of X in Theorem 3.5 (i) does not depend on the approximating sequence θ_k because if we have two sequences $\{\theta_k\}$ and $\{\bar{\theta}_k\}$ converging to θ then we can apply the theorem to the sequence $\theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2, \dots$

(ii) Suppose that $z_0 \in D_*$, $\mu_0 \in \mathbb{R}$, and h is positive and harmonic in D_* with $h(0) = 1/\pi$. By Theorem 2.1, we can find $\theta \in \mathcal{T} \leftrightarrow (h, \mu_0) \in \mathcal{H}$. Let X be the process corresponding to z_0 and θ as in Theorem 3.5. Then X has a stationary distribution with the density h and μ_0 is the rate of rotation of X in the sense of Theorem 3.5 (v)-(vi).

(iii) Theorem 3.5 establishes existence of ORBM for all angles θ of oblique reflection. ORBMs can be uniquely parametrized either by $\theta \in \mathcal{T}$ or by pairs $(h, \mu_0) \in \mathcal{H}$. We will write $X \leftrightarrow \theta$ or $X \leftrightarrow (h, \mu_0)$.

(iv) If $\theta = \pi/2$ a.e. on an open arc $I \subset \partial D_*$ then as in the proof of Proposition 2.11, $\theta + i\widetilde{\theta}$ extends to be analytic across I, and hence so does $G = e^{i(\theta + i\widetilde{\theta})}$. In this case, for $x \in I$,

$$\lim_{r\to 1} \frac{e^{-\widetilde{\theta}(rx)} \cos\theta(rx)}{r-1} = \operatorname{Re} \lim_{r\to 1} \frac{G(rx) - G(x)}{rx - x} x = \operatorname{Re} G'(x)x. \tag{3.14}$$

Thus the integral in (3.13) is finite for each $x \in I$. A similar statement holds if $\theta = -\pi/2$ a.e. on I.

Note that the process X itself will not hit a fixed point $x \in I$. The reason is that X has only a countable number of excursions from the boundary of ∂D_* and the distribution of the location of the endpoint of an excursion has a density. Hence, with probability 1, no excursion will end at x. If an excursion ends at a point in I, the process X will jump at that time to an end of the interval where $\theta = \pi/2$ a.e. Thus, X itself will avoid x forever but the same argument shows that $x \in \Gamma_X^\theta$ with probability 1 because Γ_X^θ contains the arcs between the endpoints of excursions hitting points inside I and the points to which X jumps at those times.

(v) Let ν be the positive measure on ∂D_* defined by $h(z) = \int_{\partial D_*} K_x(z)\nu(dx)$, where $K_x(z)$ is the Poisson kernel for $z \in D_*$. Fix $x \in \partial D_*$ and write

$$h(rx) = c\frac{1+r}{1-r} + \int_{\partial D_*} \frac{1-r^2}{|y - rx|^2} d\sigma(y)$$

where σ is a positive measure with $\sigma(\{x\}) = 0$ and $c = \nu(\{x\})$. Then

$$\lim_{r\to 1} (1-r)h(rx) = 2c \tag{3.15}$$

as can be seen by splitting the integral into $\int_I + \int_{\partial D_* \setminus I}$ where $x \in I$ and $\sigma(I) < \varepsilon$. If $c = \nu(\{x\}) > 0$, then

$$\int_0^1 \frac{h(rx)}{h(rx)^2 + (\widetilde{h}(rx) - \mu_0/\pi)^2} \frac{dr}{1-r} \le \int_0^1 \frac{1}{(1-r)h(rx)} dr < \infty.$$

and so $x \in \Gamma_X^\theta$ with probability 1 by Theorem 3.5 (vii), where $X \leftrightarrow (h, \mu_0)$.

(vi) The condition $\nu(\{x\}) > 0$ is stronger than the integrability condition (3.13). For example, if $h(z) = \frac{1}{\pi}\operatorname{Re}(1-z)^{-p}$, with $0 < p < 1$, then $\int_0^1 \frac{1}{(1-r)h(r)} dr < \infty$ so that (3.13) holds at $x = 1$. However, by (3.15), the corresponding positive measure ν satisfies $\nu(\{1\}) = 0$.

(vii) Suppose $\mu_0 = 0$. If $h(x) = h(\overline{x})$ for all $x \in \partial D_*$, where $\overline{x}$ denotes the complex conjugate of x, then $\widetilde{h}(r) = 0$ for $-1 < r < 1$. In this case, the integral in (3.13) is finite for $x = 1$ if and only if

$$\int_0^1 \frac{1}{(1-r)h(r)} dr < \infty. \tag{3.16}$$

Condition (3.13) can be restated. Set $f = \operatorname{Re}(1/(h + i\widetilde{h} - i\mu_0/\pi))$. Then f is harmonic and positive, so there is a positive measure $d\sigma$ such that

$$f(z) = \int_{\partial D_*} \frac{1-|z|^2}{|w-z|^2} d\sigma(w).$$

PROPOSITION 3.7. *Condition* (3.13) *holds for* $x \in \partial D_*$ *if and only if*

$$\int_{\partial D_*} \frac{1}{|w-x|} d\sigma(w) < \infty. \tag{3.17}$$

For example, suppose E is a closed subset of ∂D_* of positive length. Let $f(y) = \mathrm{dist}(y, E)^p$ for $y \in \partial D_*$, where $p \in (0,1)$ is fixed. Then it is not hard to verify that $f \in C^p(\partial D_*)$, that is, f is Hölder-continuous with exponent p on ∂D_*. Let the harmonic extension of f to D_* be also denoted by f. Thus the function $f + i\widetilde{f}$ is analytic on D_*, extends to be continuous on the closed disk $\overline{D}_*$, and hence the zero set $Z = \{y \in \partial D_* : f(y) = \widetilde{f}(y) = 0\} \subset E$ has zero length (see (Hoffman, 1962, page 51)). Set $h + i\widetilde{h} = 1/(f + i\widetilde{f})$. Then h is positive and harmonic on D_*. Since $f \in C^p(\partial D_*)$, $\widetilde{f} \in C^p(\partial D_*)$ by Theorem II.3.2 in Garnett and Marshall (2005). Thus $h = f/(f^2 + \widetilde{f}^2)$ is continuous up to $\partial D_* \setminus Z$, and so h tends to 0 as $z \to E \setminus Z$. The function h tends to a positive number at each point of $\partial D_* \setminus E$. The positive measure $\sigma(dy) = f(y)|dy|$ on ∂D_* satisfies (3.17) for each $x \in E$, since $f(y) \le |x-y|^p$ for every $x \in E$. Let $\theta \in \mathcal{T} \leftrightarrow (h, 0) \in \mathcal{H}$ and $X \leftrightarrow (h, 0)$ be the corresponding ORBM. By Theorem 3.5 (vii) and Proposition 3.7, for every $x \in E$, $x \in \Gamma_X^\theta$ with probability 1. Note also that the integral in (3.17) is infinite for each point $x \in \partial D_* \setminus E$, since f is positive and continuous there. So for every $x \in \partial D_* \setminus E$, this ORBM does not hit x with probability 1. The function θ is continuous on $\partial D_* \setminus Z$, and $|\theta| < \pi/2$ off E. We can take E to have no interior in ∂D_*, so $|\theta| < \pi/2$ on a dense open set.

Recall that if $f : D_* \to D_*$ is a conformal map of D_* onto itself, then there exist $\theta_0 \in [0, 2\pi)$ and $w_0 \in D_*$ such that $f(z) = e^{i\theta_0}\frac{z-w_0}{1-\overline{w_0}z}$. So in particular f extends continuously to $\overline{D}_*$ as a smooth homeomorphism. The following result establishes conformal invariance of ORBM on the unit disk.

THEOREM 3.8. *Suppose* $\theta \in \mathcal{T}$ *and* X *is an ORBM on* D_* *with reflection angle* θ. *Suppose* $f : D_* \to D_*$ *is a conformal map of* D_* *onto* D_*. *Define for* $t \in [0, \infty)$,

$$c(t) = \int_0^t |f'(X_s)|^2 ds \quad \text{and} \quad Y_t = f(X_{c^{-1}(t)}). \tag{3.18}$$

Then Y *is an ORBM on* D_* *with reflection angle* $\theta \circ f^{-1} \in \mathcal{T}$. *Equivalently, if* $(h, \mu_0) \in \mathcal{H} \leftrightarrow \theta$, *then* Y *is the ORBM on* D_* *parametrized by* $(\bar{h}, \bar{\mu}_0) \in \mathcal{H}$, *where* $\bar{h}(z) = h(f^{-1}(z))/(\pi h(f^{-1}(0)))$ *and* $\bar{\mu}_0 = \mu(f^{-1}(z))/h(f^{-1}(0))$. *Here* $\mu(w)$ *is the harmonic function defined by* (2.24).

3.3. *Excursion Reflected Brownian Motions.* We now address the question that was left unanswered in Section 3.2, namely whether there exists a process on D_* associated with a purely tangential angle of reflection, e.g., $\theta \equiv \pi/2$. In Theorem 3.12 we will show that such a process does indeed exist and can be obtained as a suitable limit of ORBMs in D_* corresponding to angles of reflection $\theta \in \mathcal{T}$. We refer to this process as excursion reflected Brownian motion (ERBM).

We will first define ERBM more generally, in a bounded simply connected domain D with variable excursion intensity $\nu(dx)$, where ν is a measure on ∂D. Our construction resembles a process introduced in Fukushima and Tanaka (2005); Chen, Fukushima and Ying (2007); Chen and Fukushima (2008) and called "Brownian motion extended by darning" (BMD), and defined simultaneously in Lawler (2006) under the name of ERBM. We will use some concepts from excursion theory reviewed in Section 2.2.

DEFINITION 3.9. Suppose that $\nu(dx)$ is a finite positive measure on ∂D. Let H^x be the standard Brownian excursion law in D for excursions starting at $x \in \partial D$. If $D = D_*$ then we normalize the σ-finite measures H^x, $x \in \partial D_*$, so that all of them can be obtained from H^1 by rotation around 0. Let $\mathbf{\Delta}$ be a cemetery state and $\mathcal{C} = \mathcal{C}_D$ denote the family of all functions $\omega : [0, \infty) \to \overline{D} \cup \{\mathbf{\Delta}\}$ such that $\omega(0) \in \partial D$, ω is continuous up to its lifetime $\zeta < \infty$, and $\omega(t) = \mathbf{\Delta}$ for $t \geq \zeta$. Let λ denote the Lebesgue measure on $\mathbb{R}_+ = [0, \infty)$ and let $\mathcal{P}$ be the Poisson point process on $\mathbb{R}_+ \times \mathcal{C}$ with characteristic measure $\lambda \times \int_{\partial D} H^x \nu(dx)$. With probability 1, there are no two points with the same first coordinate so the elements of $\mathcal{P}$ may be unambiguously denoted by (t, e_t). Let

$$\zeta_t = \inf\{s > 0 : \mathrm{e}_t(s) = \mathbf{\Delta}\}.$$

Let $\sigma_v = \sum_{s \leq v} \zeta_s$ and $\sigma_{v-} = \sum_{u < v} \zeta_u$ for $v \geq 0$.

Let $D^\partial := D \cup \{\partial\}$ be a one-point compactification of D obtained by identifying the usual boundary ∂D with a single point ∂.

If $D = D_*$ then the lifetimes of excursions of the process $\mathcal{P}$ have the same structure as those of the symmetric reflected Brownian motion (with the normal reflection), so $\sigma_v < \infty$ for all $v < \infty$ and $\lim_{v \to \infty} \sigma_v = \infty$, a.s. For all domains D for which the last two statements are true, with probability 1, for every $t \geq 0$, the formula $r = \inf\{v \geq 0 : \sigma_v \geq t\}$ defines a unique $r \geq 0$. For $t \geq 0$ let

$$X_t = \begin{cases} \mathrm{e}_r(t - \sigma_{r-}), & \text{if } \sigma_{r-} < \sigma_r \text{ and } t \in [\sigma_{r-}, \sigma_r), \\ \partial, & \text{otherwise.} \end{cases}$$

With probability one, X is a conservative process taking values in D^∂. We will call the process X (or its distribution) excursion reflected Brownian motion (ERBM) in D with excursion intensity ν. In general, X is not a Hunt process on $\overline{D}$ as it does not have the quasi-left continuity property at the first hitting time of ∂D, which is a predictable stopping time. However, X is a conservative continuous Hunt process on D^∂.

REMARK 3.10. (i) If H^x is a standard Brownian excursion law in D and $c > 0$ is a constant then cH^x is also a standard Brownian excursion law in D. We talked about "the" standard excursion laws above because all standard excursion laws in a simply connected domain corresponding to a given boundary point are constant multiples of each other.

(ii) For any strictly positive function $a(x)$ on the boundary of D, ERBM corresponding to $(a(x)\nu(dx), (1/a(x))H^x)_{x \in \partial D}$ has the same distribution as ERBM determined by $(\nu(dx), H^x)_{x \in \partial D}$. Hence, one has to specify both ν and the normalization of the excursion laws H^x to identify ERBM uniquely.

(iii) It may be surprising at the first sight but it is easy to see that for any constant $c > 0$, $(\nu(dx), H^x)_{x \in \partial D}$ and $(c\nu(dx), H^x)_{x \in \partial D}$ define the same ERBM. So we may assume that ν is a probability measure.

(iv) Combining the last two remarks, it is easy to check that if ERBM X can be represented by $(\nu(dx), H^x)_{x\in\partial D}$ and also by $(\nu_1(dx), H_1^x)_{x\in\partial D}$, then

$$(\nu_1(dx), H_1^x)_{x\in\partial D} \equiv (ca(x)\nu(dx), (1/a(x))H^x)_{x\in\partial D}$$

for some positive function $a(x)$ and some positive constant c.

(v) When D is the unit ball D_*, the ERBM in D_* with excursion intensity ν being the uniform measure on ∂D_* has the same distribution as the BMD studied in Fukushima and Tanaka (2005); Chen, Fukushima and Ying (2007); Chen and Fukushima (2008); see (Chen and Fukushima, 2012, Remark 7.6.4) where this identification is proved when D is the exterior of the unit ball. When D is the exterior of the unit ball, the process also has the same distribution as the ERBM introduced in Lawler (2006); see (Chen and Fukushima, 2015, Example 6.3).

(vi) To make things simple, we will assume in theorems on ERBM that ∂D is a Jordan curve (in other words, D is a simply connected Jordan domain). This is equivalent to saying that if $f : D_* \to D$ is a one-to-one and onto analytic mapping then f can be extended to be continuous and one-to-one on $\overline{D}_*$. We believe that all our results hold for arbitrary bounded simply connected domains because "exotic" points on the boundary are negligible from the point of view of excursion theory.

(vii) The reader who wishes to learn more about potential theoretic properties of domains and their relationship to geometric properties may consult Ohtsuka (1970) for a discussion of "prime ends." The Martin boundary is presented in Doob (1984); in particular, the identification of the Martin boundary and prime ends is mentioned in (Doob, 1984, 1 XII 3). The Martin topology and boundary in simply connected planar domains are conformally invariant, see (Pommerenke, 1975, Thm. 9.6).

(viii) If D is a Jordan domain and $x \in \partial D$, then the Martin kernel $K_x(\,\cdot\,)$ is the unique, up to a multiplicative constant, positive harmonic function in D that vanishes everywhere on the boundary except at x. The density of the expected occupation measure for H^x is a constant multiple of the Martin kernel $K_x(\,\cdot\,)$ by (Burdzy, 1987, Prop. 3.4).

PROPOSITION 3.11. *Suppose $D \subset \mathbb{C}$ is a bounded simply connected Jordan domain.*

(i) *Let X be an ERBM constructed from $(\nu, H^x)_{x\in D}$, where ν is a probability measure on ∂D. Then X has a unique stationary distribution whose density is proportional to $h(y) = \int_{\partial D} K_x(y)\nu(dx)$.*

(ii) *For every positive harmonic function h in D with $\|h\|_{L^1(D)} = 1$ there exists an ERBM X with the stationary density h.*

We say that a real-valued function f defined on a subset S of $\mathbb{R}^n$ is Lipschitz with constant $\lambda < \infty$ if $|f(x) - f(y)| \le \lambda|x - y|$ for all $x, y \in S$. It follows from the definitions that a Lipschitz function is Dini continuous.

THEOREM 3.12. (i) *Consider a sequence of C^2 functions $\theta_k : \partial D_* \to (-\pi/2, \pi/2)$ and let X^k be defined by*

$$X_t^k = x_k + B_t + \int_0^t \mathbf{v}_{\theta_k}(X_s^k) dL_s^k, \qquad \textit{for } t \ge 0. \tag{3.19}$$

Let $(h_k, \mu_{0,k}) \leftrightarrow \theta_k$ as in Lemma 2.2. We make the following assumptions:

(a) *θ_k converge to $\pi/2$ almost everywhere.*
(b) *For some $c_1 > -\pi/2$ and all x and k, $\theta_k(x) \geq c_1$.*
(c) *There exist $\lambda < \infty$ and $c_2 > 0$ such that h_k restricted to ∂D_* is Lipschitz with constant λ for every k, and $h_k(x) > c_2$ for every x and k.*
(d) *There is a finite measure $\nu(dx)$ on ∂D_* such that $h_k(x)dx \to \nu(dx)$ weakly as measures on ∂D_*, when $k \to \infty$.*
(e) $\lim_{k\to\infty} \operatorname{dist}(x_k, \partial D_*) = 0$.

Then the processes X^k converge in the sense of finite dimensional distributions to ERBM X corresponding to $(\nu(dx), H^x)_{x\in\partial D_*}$, where all H^x are obtained from H^1 by rotation around 0.

(ii) *Conversely, suppose that h is harmonic in D_*, Lipschitz on $\overline{D}_*$ and positive on $\overline{D}_*$. Then there exists a sequence of C^2 functions $\theta_k : \partial D_* \to (-\pi/2, \pi/2)$ satisfying conditions (a)-(e) with $\nu(dx) = h(x)dx$ on ∂D_*. ORBMs X^k corresponding to θ_k's converge in the sense of finite dimensional distributions to an ERBM X with the stationary density h.*

REMARK 3.13. (i) The roles of $\pi/2$ and $-\pi/2$ in Theorem 3.12 can be reversed by replacing $\theta_k(x)$ with $-\theta_k(\overline{x})$. See Remark 2.6(iv).

(ii) It is easy to see from Theorems 3.8 and 3.12 that if $f : D_* \to D_*$ is a conformal map and X is an ERBM on D_* corresponding to $(\nu(dx), H^x)_{x\in\partial D_*}$, then $f(X)$ is a time-change of ERBM on D_* corresponding to $(\nu(dx) \circ f^{-1}, H^x)_{x\in\partial D_*}$.

(iii) Suppose that there exists $\lambda < \infty$ such that h_k restricted to ∂D_* is Lipschitz with constant λ for every k. Then it is elementary to show that there exists $c_2 > 0$ such that $h_k(x) > c_2$ for every x and k if and only if there exists $\lambda_1 < \infty$ such that $1/h_k$ restricted to ∂D_* is Lipschitz with constant λ_1 for every k.

EXAMPLE 3.14. Theorem 3.12 has many assumptions so it deserves a simple example to illustrate it. Suppose $h(x)$ and $1/h(x)$ are positive Lipschitz continuous functions on ∂D_* with $\|h\|_{L^1(D_*)} = 1$. Let $h(z)$ be the harmonic extension of h to D_*. Suppose also that $\mu_{0,k} \to \infty$ as $k \to \infty$. Then $(h, \mu_{0,k}) \leftrightarrow \theta_k \in \mathcal{T}$ as in Theorem 2.1. If $h_k = h$ for all k then $(h_k, \mu_{0,k})$ and θ_k satisfy assumptions (a)-(e) of Theorem 3.12.

3.4. *ORBMs in Simply Connected Domains.* We will use conformal mappings to construct ORBMs in arbitrary simply connected domains. In the following, we will usually use X to denote ORBM in the disk D_* and Y to denote ORBM in other domains.

THEOREM 3.15. *Suppose that f is a one-to-one analytic function mapping D_* onto a simply connected domain $D \subset \mathbb{C}$. Suppose that $\theta \in \mathcal{T}$, $\theta \leftrightarrow (h, \mu)$, let $\bar{h} = h \circ f^{-1}$ and assume that $\bar{h}$ is in $L^1(D)$. Let $X \leftrightarrow \theta$ be ORBM in D_* and define*

$$c(t) = \int_0^t |f'(X_s)|^2 ds, \qquad \text{for } t \geq 0, \tag{3.20}$$

$$\zeta = \inf\{t \geq 0 : c(t) = \infty\}, \tag{3.21}$$

$$Y_t = f(X_{c^{-1}(t)}), \qquad \text{for } t \in [0, \zeta). \tag{3.22}$$

We will call Y an ORBM in D. The following hold.

(i) *With probability 1, $\zeta = \infty$.*

(ii) *The process Y is an extension of killed Brownian motion in D in the sense that for every $t \ge 0$ and $\tau_t = \inf\{s \ge t : Y_s \in \partial D\}$, the process $\{Y_s, s \in [t, \tau_t)\}$ is Brownian motion killed upon exiting D.*

(iii) *The process Y has a stationary distribution with density $\widehat{h} = \bar{h}/\|\bar{h}\|_{L^1(D)}$.*

(iv) *Recall that μ is the function given by (2.24). For $z \in D$, let $\mathbf{arg}^*(Y_t - z) = \mathbf{arg}^*(X_{c^{-1}(t)} - f^{-1}(z))$ for all t. Then, for every $z \in D$, a.s.*

$$\lim_{t\to\infty} \frac{\mathbf{arg}^*(Y_t - z)}{t} = \frac{\mu(f^{-1}(z))}{\|\bar{h}\|_{L^1(D)}}. \tag{3.23}$$

(v) *Suppose that $\mu_0 \in \mathbb{R}$ and $\widehat{h}$ is a positive harmonic function in D with $\|\widehat{h}\|_{L^1(D)} = 1$. Then there exists an ORBM Y in D with the following properties.*
(a) *The stationary distribution of Y is $\widehat{h}(x)dx$.*
(b) *Set $g = f^{-1}$ and define*

$$b(t) := \int_0^t |(g'(Y_s)|^2 ds, \qquad t \ge 0, \tag{3.24}$$

$$X_t := g(Y_{b^{-1}(t)}), \qquad t \ge 0, \tag{3.25}$$

$$\mathbf{arg}^* Y_t := \mathbf{arg}^* X_{b(t)}, \qquad t \ge 0. \tag{3.26}$$

Since $\widehat{h} \circ f$ is a positive harmonic function on D_*, $\|\widehat{h} \circ f\|_1 = \pi\widehat{h} \circ f(0) < \infty$. Set $h_1 = \widehat{h} \circ f/\|\widehat{h} \circ f\|_1$ and let $\mu \in \mathcal{R} \leftrightarrow (h_1, \mu_0) \in \mathcal{H}$. Then X is the ORBM in D_* parametrized by (h_1, μ_0) and (3.23) holds with $\bar{h} = h_1 \circ f^{-1} = \widehat{h}/\|\widehat{h} \circ f\|_1$.

(vi) *(Consistence) If D has a smooth boundary and θ is C^2 then the distribution of Y is the same as that of the process identified in Theorem 3.1 (ii) relative to $\theta \circ f^{-1}$.*

REMARK 3.16. (i) The quantity $\mathbf{arg}(Y_t - z)$ has a natural interpretation when Y is continuous, namely, $\mathbf{arg}(Y_t - z) - \mathbf{arg}(Y_0 - z)$ is the number of windings of Y around z over the time interval $[0, t]$. The quantity $\mathbf{arg}^*(Y_t - z)$ is obtained from $\mathbf{arg}(Y_t - z)$ by discarding (the windings of) all excursions of Y which make more than a full loop around z (from endpoint to endpoint of the excursion, not within the excursion). Our definition of $\mathcal{E}^L_s$ was chosen to make this simple geometric interpretation of $\mathbf{arg}^*(Y_t - z)$ possible.

Unfortunately, when Y is not continuous, $\mathbf{arg}^*(Y_t - z)$ does not have a simple intuitive interpretation because the definition of $\mathbf{arg}$ in D_* depends on θ.

(ii) The process Y constructed in Theorem 3.15 will be called ORBM in D. The family of ORBMs in D can be parametrized either in terms of pairs (θ, f) or triplets $(\widehat{h}, \mu_0, f)$, so we will write $Y \leftrightarrow (\theta, f)$ or $Y \leftrightarrow (\widehat{h}, \mu_0, f)$. The function f provides a way to parametrize ∂D, in a sense.

(iii) If $\mu \in \mathcal{R} \leftrightarrow (h, \mu_0) \in \mathcal{H}$ then we say that $\mu \circ f^{-1}(z)$ is the rotation rate about $z \in D$ for the process Y given by (3.22). If μ_1 is a harmonic function defined on D, let $\widetilde{\mu}_1$ be the harmonic conjugate of μ_1 vanishing at $f(0)$. Then $\widetilde{\mu_1 \circ f}$ is a harmonic function on D_* vanishing at 0 and $\widetilde{\mu_1 \circ f} = \widetilde{\mu_1} \circ f$. By Theorem 2.1 and Theorem 3.15, μ_1 is a rotation rate for an ORBM if and only if $\widetilde{\mu}_1(z) > -1$ for all $z \in D$.

(iv) Suppose that f is a conformal mapping from a bounded simply connected planar domain D_1 to another bounded simply connected planar domain D_2. Let $\mathcal{K}(D_1, D_2, f)$ be the family of positive integrable harmonic functions h in D_1 such that $h \circ f^{-1} \in L^1(D_2)$. By

Theorems 3.8 and 3.15, f establishes a correspondence between a subfamily of ORBMs on D_1 that have the density of stationary distribution in $\mathcal{K}(D_1, D_2, f)$ and a subfamily of ORBMs on D_2 that have the density of stationary distribution in $\mathcal{K}(D_2, D_1, f^{-1})$. The subfamilies are non-empty because they always contain normally reflected Brownian motions. Theorem 3.20 below gives some sufficient conditions on the integrability of positive harmonic functions in domains. The correspondence between ORBMs on different planar domains need not extend to all ORBMs on either side because the assumption $\bar{h} \in L^1(D)$ of Theorem 3.15 does not hold for some h and f; see Example 4.1 below.

(v) There exist processes in D that are extensions of Brownian motion in D, which have a stationary density and a "limiting rate of rotation" μ_0 and which are not ORBM's. An example of such a process is the conformal image of reflected Brownian motion in D_* with diffusion on the boundary (see a Ph.D. thesis Card (2009) devoted to this class of processes).

THEOREM 3.17. *Suppose that $D \subset \mathbb{C}$ is a simply connected bounded Jordan domain and f is a conformal mapping from D_* onto D, which, by Carathéodory's theorem, necessarily extends to a homeomorphism from $\overline{D}_*$ onto $\overline{D}$. Consider a sequence of C^2 functions $\theta_k : \partial D_* \to (-\pi/2, \pi/2)$ and processes X^k which satisfy* (3.19) *and assumptions (a)-(e) of Theorem 3.12. Let $(h_k, \mu_k) \leftrightarrow \theta_k$ and let $c_k(t), \zeta_k$ and Y^k be defined relative to θ_k, f and X^k as in Theorem 3.15.*

Let ν, h and X be defined as in Theorem 3.12. Let $c(t), \zeta$ and Y be defined relative to θ, f and X as in Theorem 3.15. In (i)-(iv) below, $\bar{h} := h \circ f^{-1}$ is assumed to be in $L^1(D)$.

(i) *Almost surely, $\zeta_k = \infty$ for every $k \geq 1$ and $\zeta = \infty$.*
(ii) *The process Y is an ERBM in D corresponding to $(\bar{\nu}(dx), \bar{H}^x)_{x \in \partial D}$ with excursion intensity $\bar{\nu}$ defined by $\bar{\nu}(A) = \nu(f^{-1}(A))$ for $A \subset \partial D$, and excursion laws $\bar{H}^x$ normalized so that the density of the expected occupation time for $\bar{H}^x$ is the Martin kernel $K_x(\cdot)$ in D normalized by $K_x(f(0)) = 1$.*
(iii) *Processes Y^k converge to Y in the sense of convergence of finite dimensional distributions.*
(iv) *The process Y has a stationary distribution with the density $\widehat{h} = \bar{h}/\|\bar{h}\|_{L^1(D)}$.*
(v) *For every positive harmonic function $\widehat{h}$ in D with $\|\widehat{h}\|_{L^1(D)} = 1$ such that $\widehat{h} \circ f$ is Lipschitz on $\overline{D}_*$ and strictly positive on ∂D_*, there is a sequence of C^2 functions $\theta_k : \partial D_* \to (-\pi/2, \pi/2)$ such that Y^k and Y can be constructed as in the initial part of the theorem and the stationary measure for ERBM Y has density $\widehat{h}$.*

The next two theorems show that ORBM in an arbitrary domain (possibly with a fractal boundary) can be approximated by ORBMs in smooth domains where the oblique angle of reflection has a natural interpretation. This provides a justification of the name "obliquely reflected Brownian motion" for processes in domains with rough boundaries.

THEOREM 3.18. *Suppose that $D \subset \mathbb{C}$ is a simply connected Jordan domain, $y_0 \in D$ and f is a conformal mapping from D_* onto D which, necessarily, has a one-to-one continuous extension to $\overline{D}_*$. Let D_k be simply connected domains with smooth boundaries such that $y_0 \in D_k \subset D_{k+1} \subset D$ for all k and $\bigcup_k D_k = D$. Let $f_k : D_* \to D_k$ be conformal mappings such that $f_k^{-1}(y_0) = f^{-1}(y_0)$ and $f_k \to f$ as $k \to \infty$.*

Suppose that $\mu_0 \in \mathbb{R}$, $\bar{h} \in L^1(D)$ is positive and harmonic with $\|\bar{h}\|_{L^1(D)} = 1$, and $\bar{h} \circ f$ is strictly positive on ∂D_*. Let Y be the process constructed as in Theorem 3.15 (v), relative to

D, f, μ_0 and $\bar{h}$, with $Y_0 = y_0$. Let $\bar{h}_k = \bar{h}/\|\bar{h}\|_{L^1(D_k)}$. Let Y^k be defined in the same way that Y was defined, relative to D_k, f_k, μ_0 and $\bar{h}_k$, with $Y_0^k = y_0$. Then Y^k converge weakly to Y in $M_1^{\mathfrak{I}}$ topology.

The following concrete example shows how one can approximate a general ORBM in D by ORBMs in an increasing sequence of smooth domains with smooth reflection angles. Suppose that $Y \leftrightarrow (\theta, f) \leftrightarrow (\bar{h}, \mu_0, f)$. Take any strictly increasing sequence of positive numbers r_k that increases to 1. Let $D_k = f(B(0, r_k))$ and $f_k(z) = f(z/r_k)$. It is easy to see that D_k is a smooth subdomain of D and f_k is a conformal mapping from $B(0, r_k)$ to D. Clearly $h_k(z) := \bar{h}(f(r_k z))$ is a positive harmonic function on D_* that is smooth on $\overline{D}_*$. By Theorem 2.1, $\theta_k \leftrightarrow (h_k/h_k(0), \mu_0)$ is smooth on ∂D_*. Thus $\bar{\theta}_k(w) = \theta_k(f^{-1}(w)/r_k) \in (-\pi/2, \pi/2)$ defines a smooth function on ∂D_k. Let Y^k be the ORBM on D_k with reflection angle $\bar{\theta}_k$ constructed in Theorem 3.1(ii). Theorem 3.18 asserts that Y^k converge weakly to ORBM Y on D in $M_1^{\mathfrak{I}}$ topology.

THEOREM 3.19. *Suppose that $D \subset \mathbb{C}$ is a simply connected Jordan domain, $y_0 \in D$ and $f : D_* \to D$ is a conformal mapping which, necessarily, has a one-to-one continuous extension to $\overline{D}_*$. Let D_k be simply connected domains with smooth boundaries such that $y_0 \in D_k \subset D_{k+1} \subset D$ for all k and $\bigcup_k D_k = D$. Let $f_k : D_* \to D_k$ be one-to-one analytic functions such that $f_k^{-1}(y_0) = f^{-1}(y_0)$ and $f_k \to f$ as $k \to \infty$.*

Suppose that $\theta : \partial D \to (-\pi/2, \pi/2)$ is a continuous function. Let $\theta_* : \partial D_* \to (-\pi/2, \pi/2)$ be defined by $\theta_* = \theta \circ f$. Let Y be ORBM in D, such that $Y \leftrightarrow (\theta_*, f)$ and $Y_0 = y_0$.

For every k, let $g_k : \partial D_k \to \partial D$ be a measurable function such that for every $x \in \partial D_k$, $g_k(x) = y \in \partial D$ and $|x - y| = \operatorname{dist}(x, \partial D)$. Let $\theta_k(x) = \theta(g_k(x))$ for $x \in \partial D_k$. Let Y^k be the ORBM in D_k such that $Y^k \leftrightarrow (\theta_k, f_k)$ and $Y_0^k = y_0$. Then Y^k's converge weakly in M_1 topology to Y.

The assumption that $\bar{h} \in L^1(D)$ applied in Theorem 3.15 is sufficient but not necessary. We will sketch an argument illustrating this claim in Example 4.2 below. In other words, the construction given in Theorem 3.15 generates a process Y_t for all $t \geq 0$ for some domains D and functions $\bar{h}$ such that $\|\bar{h}\|_{L^1(D)} = \infty$. Of course, in such a case no constant multiple of $\bar{h}(x)dx$ can be the stationary (probability) distribution for Y, although it can be an invariant measure.

In view of the assumption of integrability of $\bar{h}$ made in Theorems 3.15 and 3.18, it would be useful to have an effective tool to check whether a given harmonic function is in $L^1(D)$. We do not have such a test and we doubt that a universal test of this kind exists. We do have some sufficient conditions for integrability of positive harmonic functions. First, recall Theorem 2.10. It contains a criterion for a harmonic function h in D_* corresponding to an angle of oblique reflection θ to be bounded. A "push" $h \circ f^{-1}$ of such function to a bounded simply connected domain is also bounded, and hence integrable. Second, Theorem 3.20 below presents some examples of domains where all positive harmonic functions are integrable.

Recall that a function $\psi : \mathbb{R} \to \mathbb{R}$ is Lipschitz, with constant $\lambda < \infty$, if $|\psi(x) - \psi(y)| \leq \lambda |x - y|$ for all $x, y \in \mathbb{R}$. A domain $D \subset \mathbb{R}^2$ is said to be Lipschitz, with constant λ, if there exists $\delta > 0$ such that, for every $x \in \partial D$, there exists an orthonormal basis (e_1, e_2) and a

Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$, with constant λ, such that

$$\{y \in \mathcal{B}(x, \delta) \cap D\} = \{y \in \mathcal{B}(x, \delta) : \psi(\langle y, e_1\rangle) < \langle y, e_2\rangle\}.$$

We recall the definition of a John domain following Aikawa (2000). Let $\delta_D(x) = \mathrm{dist}(x, \partial D)$ and $x_0 \in D$. We say that D is a John domain with John constant $c_J > 0$ if each $x \in D$ can be joined to x_0 by a rectifiable curve γ such that $\delta_D(y) \geq c_J \ell(\gamma(x, y))$ for all $y \in \gamma$, where $\gamma(x, y)$ is the subarc of γ from x to y and $\ell(\gamma(x, y))$ is the length of $\gamma(x, y)$. The first two parts of the following theorem follow from Theorems 1 and 2 of Aikawa (2000).

THEOREM 3.20. (i) ((Aikawa, 2000, Thm. 1)) *If $D \subset \mathbb{R}^2$ is a bounded John domain with John constant $c_J \geq 7/8$ then all positive harmonic functions in D are in $L^1(D)$.*
(ii) ((Aikawa, 2000, Thm. 2)) *If $D \subset \mathbb{R}^2$ is a bounded Lipschitz domain with constant $\lambda < 1$ then all positive harmonic functions in D are in $L^1(D)$.*
(iii) *There exists a bounded Lipschitz domain D with constant $\lambda = 1$ and a positive harmonic function h in D which is not in $L^1(D)$.*

4. Proofs.

Proof of Theorem 3.1. (i) This part is a special case of (Harrison, Landau and Shepp, 1985, Thm. 2.6).

(ii) Let X be the unique pathwise solution of (2.1). Then by Itô's formula, $f(X_t) - \frac{1}{2}\int_0^t \Delta f(X_s)ds$ is a submartingale under $\mathbb{P}_z$ for every $z \in \overline{D}$ and $f \in \mathcal{C}$. Thus, in view of (i), $(X, \mathbb{P}_z)$ is the unique solution to the submartingale problem (2.3).

(iii) This part is known, see, e.g., Kim, Kim and Yun (1998). For the reader's convenience, we give a sketch of the Dirichlet form approach to the construction of ORBM. The argument given below works in higher dimensions as well. In C^2-smooth domains with C^2-smooth reflection angle, it is enough to construct ORBM locally nearly the boundary and then patch the pieces together. Thus by locally flattening the boundary, we may and do assume that $D = \mathbb{H}$, the upper half space. Let $\mathbf{v}(x) = (v_1(x), 1)$ for $x \in \partial\mathbb{H}$ with $v_1(x) := \tan\theta(x)$. Consider a non-symmetric bilinear form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{H}, dz)$, where

$$\begin{aligned}
\mathcal{F} &= \left\{f \in L^2(\mathbb{H}, dz) : \nabla f \in L^2(\mathbb{H}, dz)\right\}, \\
\mathcal{E}(f, g) &= \int_{\mathbb{H}} \nabla f(z) \cdot \nabla g(z) dz - \int_{\partial\mathbb{H}} v_1(x) \frac{\partial f(x, 0)}{\partial x} g(x, 0) dx \quad \text{for } f, g \in \mathcal{F}.
\end{aligned}$$

Let $\mathcal{E}^0(f, g) = \int_{\mathbb{H}} \nabla f(z) \cdot \nabla g(z) dz$, and for $\alpha > 0$,

$$\mathcal{E}^0_\alpha(f, g) := \mathcal{E}^0(f, g)) + \alpha(f, g)_{L^2(\mathbb{H};dz)} \quad \text{and} \quad \mathcal{E}_\alpha(f, g) := \mathcal{E}(f, g)) + \alpha(f, g)_{L^2(\mathbb{H};dz)}.$$

Observe that for $f \in C^2_c(\bar{\mathbb{H}})$, by the integration by parts formula,

$$\left|\int_{\partial\mathbb{H}} v_1(x) \frac{\partial f(x, 0)}{\partial x} f(x, 0) dx\right| = \frac{1}{2}\left|\int_{\partial\mathbb{H}} v_1'(x) f(0, x)^2 dx\right| \leq \frac{1}{2}\|v_1'\|_\infty \|f(x, u)\|^2_{L^2(\partial\mathbb{H}, dx)}.$$

By the boundary trace theorem, for every $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$\|f(x, u)\|^2_{L^2(\partial\mathbb{H}, dx)} \leq \varepsilon\mathcal{E}^0(f, f) + C_\varepsilon\|f\|^2_{L^2(\mathbb{H};dz)} \quad \text{for } f \in \mathcal{F}.$$

It follows from the above two displays that there are constants $\alpha > 0$ and $C_0 \geq 1$ such that

$$C_0^{-1}\mathcal{E}_1^0(f,f) \leq \mathcal{E}_\alpha(f,f) \leq C_0\mathcal{E}_1^0(f,f)$$

for every $f \in C_c^2(\overline{\mathbb{H}})$ and hence for every $f \in \mathcal{F}$. On the other hand, for $f, g \in C_c^2(\bar{\mathbb{H}})$,

$$\begin{aligned}
& -\int_{\partial\mathbb{H}} v_1(x)\frac{\partial f(x,0)}{\partial x}g(x,0)dx \\
= \;& -\int_{\partial\mathbb{H}} v_1(x)\int_0^\infty \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}g(x,y)\right)dydx \\
= \;& -\int_{\mathbb{H}} v_1(x)\frac{\partial f(x,y)}{\partial x}\frac{\partial g(x,y)}{\partial y}dydx - \int_{\mathbb{H}} v_1(x)g(x,y)\frac{\partial^2 f(x,y)}{\partial x\partial y}dydx \\
= \;& \int_{\mathbb{H}} v_1(x)\left(\frac{\partial f(x,y)}{\partial x}\frac{\partial g(x,y)}{\partial y} - \frac{\partial f(x,y)}{\partial y}\frac{\partial g(x,y)}{\partial x}\right)dydx \\
& -\int_{\mathbb{H}} v_1'(x)g(x,y)\frac{\partial g(x,y)}{\partial y}dydx. \qquad (4.1)
\end{aligned}$$

Thus, with $C_1 = 2\|v\|_\infty, +\|v'\|_\infty$,

$$\left|\int_{\partial\mathbb{H}} v_1(x)\frac{\partial f(x,0)}{\partial x}g(x,0)dx\right| \leq C_1\mathcal{E}_1^0(f,f)^{1/2}\mathcal{E}_1^0(g,g)^{1/2} \quad \text{for } f,g \in C_c^2(\bar{\mathbb{H}}).$$

Hence, the bilinear form $(\mathcal{E},\mathcal{F})$ satisfies the sector condition: there is a constant $C_2 \geq 1$ such that

$$|\mathcal{E}(f,g)| \leq C_2\mathcal{E}_\alpha(f,f)^{1/2}\mathcal{E}_\alpha(g,g)^{1/2} \quad \text{for } f,g \in \mathcal{F}.$$

Moreover, by increasing the value of α if needed, we have from (4.1) that for every $f \in C_c^2(\bar{\mathbb{H}})$,

$$\mathcal{E}(f, f-(0\vee f)\wedge 1) \geq 0 \qquad \text{and} \qquad \mathcal{E}_\alpha(f-(0\vee f)\wedge 1, f) \geq 0.$$

Thus $(\mathcal{E},\mathcal{F})$ is a regular non-symmetric Dirichlet form on $L^2(\bar{\mathbb{H}};dz)$. Let X be the Hunt process on $\bar{\mathbb{H}}$ associated with $(\mathcal{E},\mathcal{F})$. Then one can use stochastic analysis for non-symmetric Dirichlet forms to show that X satisfies the SDE (2.1) for quasi-every starting point $x \in \bar{\mathbb{H}}$ (see Kim, Kim and Yun (1998)). Since X behaves like Brownian motion inside $\mathbb{H}$, we can refine the result to allow X to start from every point $x \in \mathbb{H}$ and conclude that (2.1) holds for such X. $\square$

Proof of Theorem 3.2. (i) This part of our theorem is a special case of (Harrison, Landau and Shepp, 1985, Thm. 2.18).

(ii) Almost sure continuity of X follows from (2.1).

Recall that we are assuming that $\theta : \partial D_* \to (-\pi/2, \pi/2)$ and $\theta \in C^2$. It follows from (Garnett and Marshall, 2005, Cor. II.3.3) that h is $C^{2-\varepsilon}$ on $\overline{D}_*$ for every $\varepsilon > 0$.

Let Q denote the probability measure on D_* with density $h(z)$. We will show that

$$\mathbb{E}_Q\left[\int_0^1 g(X_s)dL_s\right] = \int_{\partial D_*} g(x)(h(x)/2)dx \qquad (4.2)$$

for every continuous function g on ∂D_*. Fix any continuous function g on ∂D_*. Its harmonic extension to $\overline{D}_*$ (also denoted g) is continuous on $\overline{D}_*$. Then for $\varepsilon \in (0,1)$,

$$\mathbb{E}_Q\left[\int_0^1 \frac{1}{\varepsilon}\mathbf{1}_{\{1-\varepsilon<|X_s|<1\}}g(X_s)ds\right] = \int_{D_*}\frac{1}{\varepsilon}\mathbf{1}_{\{1-\varepsilon<|z|<1\}}g(z)h(z)dz. \tag{4.3}$$

By continuity and boundedness of g and h, the limit of the right hand side, as $\varepsilon \to 0$, is equal to $\int_{\partial D_*} g(x)h(x)dx$. It is standard to show that $\int_0^1 \frac{1}{\varepsilon}\mathbf{1}_{\{1-\varepsilon<|X_s|<1\}}g(X_s)ds$ converges to $2\int_0^1 g(X_s)dL_s$ in distribution as $\varepsilon \to 0$. We claim that the family

$$\left\{\int_0^1 \frac{1}{\varepsilon}\mathbf{1}_{\{1-\varepsilon<|X_s|<1\}}g(X_s)ds, \varepsilon \in (0,1/2)\right\} \tag{4.4}$$

is uniformly integrable. Since g is bounded, it suffices to prove uniform integrability of the family $\left\{\int_0^1 \frac{1}{\varepsilon}\mathbf{1}_{\{1-\varepsilon<|X_s|<1\}}ds, \varepsilon \in (0,1/2)\right\}$. If we denote by $\mathcal{L}_t^a$ the local time of the two-dimensional Bessel process on $[0,1]$ reflected at 1, then the distribution of $\int_0^1 \frac{1}{\varepsilon}\mathbf{1}_{\{1-\varepsilon<|X_s|<1\}}ds$ is the same as $\frac{1}{\varepsilon}\int_{1-\varepsilon}^1 \mathcal{L}_1^a da$. The last random variable is stochastically majorized by $\sup\{\mathcal{L}_1^a : a \in [1/2,1]\}$ for every $\varepsilon \in (0,1/2)$. A version of the Trotter and Ray-Knight theorems shows that $\mathcal{L}_1^a$ is a diffusion in a, so $\sup\{\mathcal{L}_1^a : a \in [1/2,1]\}$ is an almost surely finite random variable. Therefore, the family in (4.4) is uniformly integrable. Taking $\varepsilon \to 0$ in (4.3) yields (4.2). It follows that the Revuz measure of L is $\frac{1}{2}h(x)dx$ on ∂D_*, relative to the invariant measure $h(z)dz$ on D_*.

We will now provide a representation of X using a map which is locally conformal. Let $D_- = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ be the left half-plane and $f(z) = \exp(z)$ the exponential function that maps D_- onto $D_* \setminus \{0\}$. For $x \in \partial D_-$ such that $f(x) = z \in \partial D_*$, define $\widehat{\mathbf{v}}(x) = i\tan\theta(z) - 1$. Note that $\widehat{\mathbf{v}}(x)$ is a periodic C^2-smooth function on ∂D_- with period $2\pi i$. Suppose that $\widehat{x}_0 \in \overline{D}_-$ and $\widehat{B}$ is a two-dimensional Brownian motion. It is known (see (Lions and Sznitman, 1984, Theorem 4.3)) that there is a pathwise unique solution $(\widehat{X},\widehat{L})$ to the following Skorokhod SDE,

$$\widehat{X}_t = \widehat{x}_0 + \widehat{B}_t + \int_0^t \widehat{\mathbf{v}}(\widehat{X}_s)d\widehat{L}_s, \tag{4.5}$$

where $\widehat{X}$ is a continuous process that takes values in $\overline{D}_-$ and $\widehat{L}$ is a continuous non-decreasing real-valued process with $\widehat{L}_0 = 0$ that increases only when $\widehat{X}_t \in \partial D_-$. The process $\widehat{X}$ is an ORBM in D_- with the oblique angle of reflection $\theta \circ f$. The Itô formula yields

$$\begin{aligned} f(\widehat{X}_t) &= f(\widehat{X}_0) + \int_0^t f'(\widehat{X}_s)d\widehat{B}_s + \int_0^t f'(\widehat{X}_s)\widehat{\mathbf{v}}(\widehat{X}_s)d\widehat{L}_s \\ &= f(\widehat{X}_s) + \int_0^t f'(\widehat{X}_s)d\widehat{B}_s + \int_0^t \mathbf{v}_\theta(f(\widehat{X}_s))d\widehat{L}_s, \end{aligned} \tag{4.6}$$

where f' is interpreted as the Jacobian of f. Let

$$c(t) = \int_0^t |f'(\widehat{X}_s)|^2 ds. \tag{4.7}$$

It is not hard to show that $c(t) < \infty$ for every $t > 0$, a.s. It follows that

$$c^{-1}(t) := \inf\{s > 0 : c(s) > t\}$$

is well defined for every $t > 0$ and the process $X_t := f(\widehat{X}_{c^{-1}(t)})$ satisfies (2.1) with Brownian motion $B_t := \int_0^{c^{-1}(t)} f'(\widehat{X}_s)d\widehat{B}_s$ and $L := \widehat{L}$. So X is an ORBM in D_* with the oblique angle of reflection θ. The exponential function $f(z) = \exp(z) : D_- \to D_*$ is neither one-to-one nor onto D_*, but it is locally conformal and maps ∂D_- onto ∂D_* so we will refer to the fact that X_t is an ORBM as conformal invariance of ORBM.

Let $\sigma_t = \inf\{s \geq 0 : L_s > t\} = \widehat{\sigma}_t = \inf\{s \geq 0 : \widehat{L}_s > t\}$, $A_t = \arg X_{\sigma_t}$ and $\widehat{A}_t = \operatorname{Im} \widehat{X}_{\widehat{\sigma}_t}$ for $t \geq 0$. Then $\widehat{A}$ and A are indistinguishable processes.

It follows from the uniqueness of the deterministic Skorohod problem that the process $\bar{X}_t := \widehat{X}_t - i\int_0^t \tan\theta(\widehat{X}_s)d\widehat{L}_s$ is a normally reflected Brownian motion in the left half-plane D_-. Hence, if we let $C_t = \operatorname{Im} \bar{X}(\widehat{\sigma}_t)$ for $t \geq 0$, then C_t is a Cauchy process with the initial value $C_0 = \operatorname{Im} \bar{X}_{\bar{S}} = \arg X_S$, where $\bar{S} := \inf\{t > 0 : \bar{X}_t \in \partial D_-\}$ and $S := \inf\{t > 0 : X_t \in \partial D_*\}$. Clearly, C_0 depends only on the initial starting point of X and is independent of the reflection angle θ. We have

$$A_t = \widehat{A}_t = C_t + \int_0^{\widehat{\sigma}_t} \tan(\theta(\widehat{X}_s))d\widehat{L}_s = C_t + \int_0^{\sigma_t} \tan(\theta(X_s))dL_s. \tag{4.8}$$

For $u \geq 0$, define

$$T_u = \inf\{t > u : X_t \in \partial D_*\}, \tag{4.9}$$

with the convention $\inf\emptyset := \infty$. We obtain from (4.8),

$$\begin{aligned}
\arg X_t &= A_{L_t} + \arg X_t - \arg X_{T_t} \\
&= C_{L_t} + \int_0^t \tan(\theta(X_s))dL_s + \arg X_t - \arg X_{T_t} \\
&= C_t + (C_{L_t} - C_t) + \int_0^t \tan(\theta(X_s))dL_s + (\arg X_t - \arg X_{T_t}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{t}\arg X_t - \mu_0 \quad &= \quad \frac{1}{t}C_t + \frac{1}{t}(C_{L_t} - C_t) + \left(\frac{1}{t}\int_0^t \tan(\theta(X_s))dL_s - \mu_0\right) \\
&\quad + \frac{1}{t}(\arg X_t - \arg X_{T_t}).
\end{aligned} \tag{4.10}$$

By (4.2), $\mathbb{E}_Q[L_1] = \int_{\partial D_*}(h(x)/2)dx = 1$. It follows from these remarks, (2.31), (4.2) with $g(x) = \tan\theta(x)$, and the limit-quotient theorem for additive functionals (see, e.g., (Revuz and Yor, 1999, Thm. X 3.12)) that for every $z \in \overline{D}_*$, $\mathbb{P}_z$-a.s.,

(4.11)
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \tan(\theta(X_s))dL_s = \mathbb{E}_Q\left[\int_0^1 \tan\theta(X_s)dL_s\right] = \int_{\partial D_*}(1/2)\tan\theta(x)h(x)dx = \mu_0,$$

$$\lim_{t\to\infty}\frac{1}{t}L_t = 1. \tag{4.12}$$

Fix an arbitrarily small $\varepsilon > 0$ and any $z \in \overline{D}_*$ and let

$$p_1(t) = \mathbb{P}_z(|\arg X_t - \arg X_{T_t}| > \varepsilon t). \tag{4.13}$$

We will argue that $p_1(t)$ is small for large t. Let $T'_u = \sup\{t \in [0,u] : X_t \in \partial D_*\}$ with the convention $\sup \emptyset = 0$. By the Markov property applied at time t and the symmetry of Brownian motion,

$$\mathbb{P}_z\left(\arg X_{T'_t} - \arg X_t > 0\right) = \mathbb{P}_z\left(\arg X_{T'_t} - \arg X_t < 0\right) = 1/2.$$

This and the Markov property applied at time t imply that

$$\mathbb{P}_z\left(|\arg X_{T'_t} - \arg X_{T_t}| > \varepsilon t\right) \geq p_1(t)/2. \tag{4.14}$$

For a fixed $u > 0$, the Cauchy process C is continuous at time u, a.s. Let $\delta > 0$ be so small that

$$\mathbb{P}\left(\sup_{1-\delta \leq u,v \leq 1+\delta} |C_u - C_v| \geq \varepsilon/2\right) < \varepsilon.$$

Then, by scaling, for any $t > 0$,

$$\mathbb{P}\left(\sup_{(1-\delta)t \leq u,v \leq (1+\delta)t} |C_u - C_v| \geq \varepsilon t/2\right) < \varepsilon. \tag{4.15}$$

By (4.12), we can find t_1 so large that for $t \geq t_1$,

$$\mathbb{P}_z(L_t \in ((1-\delta)t, (1+\delta)t)) \geq 1 - \varepsilon. \tag{4.16}$$

The jumps of A have the same size as those of C and occur at the same time because the last integral in (4.8) is a continuous function of t. If the events in (4.14) and (4.16) occur then C has a jump of size greater than εt at a time $s = L_t \in ((1-\delta)t, (1+\delta)t)$. The probability of this event is greater than $p_1(t)/2 - \varepsilon$, by (4.14) and (4.16). However, by (4.15), this probability is less than ε. Hence, $p_1(t)/2 < 2\varepsilon$ and, therefore, $p_1(t) < 4\varepsilon$ for sufficiently large t. This and (4.13) imply that for sufficiently large t,

$$\mathbb{P}_z\left(\frac{1}{t}|\arg X_t - \arg X_{T_t}| > \varepsilon\right) < 4\varepsilon. \tag{4.17}$$

Another consequence of (4.15) and (4.16) is that $|C_{L_t} - C_t| \leq \varepsilon t$ with probability greater than $1 - 2\varepsilon$ for large t. Thus, for sufficiently large t,

$$\mathbb{P}_z\left(\frac{1}{t}|C_{L_t} - C_t| > \varepsilon\right) < 2\varepsilon. \tag{4.18}$$

It follows from (4.11) that for sufficiently large t,

$$\mathbb{P}_z\left(\left|\frac{1}{t}\int_0^t \tan(\theta(X_s))dL_s - \mu_0\right| > \varepsilon\right) < \varepsilon. \tag{4.19}$$

Note that $(C_t - C_0)/t$ has the Cauchy distribution. Since $\varepsilon > 0$ is arbitrarily small, the last observation, (4.10), (4.17), (4.18) and (4.19) imply that the distributions of $\frac{1}{t}\arg X_t - \mu_0$ converge to the Cauchy distribution as $t \to \infty$.

(iii) Consider a modification of the process C which is left continuous with right limits. For $t \geq 0$, let

$$\Lambda_t = \sum_{s \leq t} (C_{t+} - C_t)\mathbf{1}_{\{|C_{t+}-C_t|>2\pi\}}, \qquad C_t^* = C_t - C_0 - \Lambda_t = C_t - \arg X_S - \Lambda_t.$$

The process C^* is a Cauchy process with jumps larger than 2π removed and starts from $C_0^* = 0$. It is elementary to see that C_t^* is a zero mean martingale and a Lévy process. Hence, the law of large numbers holds for C^*, that is, a.s.,

$$\lim_{t\to\infty} C_t^*/t = 0. \tag{4.20}$$

Note that the jumps removed from C correspond to increments of $\arg X$ in the sum on the right hand side of (3.1). Thus

$$\arg^* X_{\sigma(t)} = C_t^* + \int_0^{\sigma(t)} \tan(\theta(X_s))dL_s + \arg X_S, \tag{4.21}$$

and

$$\frac{1}{t}\arg^* X_{\sigma(t)} = \frac{1}{t}C_t^* + \frac{1}{t}\int_0^{\sigma(t)} \tan(\theta(X_s))dL_s + \frac{1}{t}\arg X_S. \tag{4.22}$$

It follows from (4.12) that, a.s.,

$$\lim_{t\to\infty} \sigma(t)/t = 1. \tag{4.23}$$

This, (4.11), (4.20) and (4.22) imply that for every $z \in \overline{D}_*$, $\mathbb{P}_z$-a.s.,

$$\lim_{t\to\infty} \frac{1}{t}\arg^* X_{\sigma(t)} = \mu_0. \tag{4.24}$$

We claim that

$$\lim_{t\to\infty} T_t/t = 1, \quad \text{a.s.} \tag{4.25}$$

First note that since $\int_0^\infty 1_{\{X_s\in D_*\}}dL_s = 0$, we have by (4.12) that $\lim_{t\to\infty} T_t = \infty$. For every $\varepsilon > 0$, $L_t - \varepsilon < L_{T_t} \leq L_t$ so

$$\frac{L_t - \varepsilon}{t} < \frac{L_{T_t}}{T_t}\frac{T_t}{t} \leq \frac{L_t}{t}.$$

This together with (4.12) establishes the claim (4.25). Combining (4.23), (4.24) and (4.25) yields

$$\lim_{t\to\infty} \frac{1}{t}\arg^* X_{T_t} = \mu_0. \tag{4.26}$$

Next we will argue that (4.26) implies that $\lim_{t\to\infty} \frac{1}{t} \arg^* X_t = \mu_0$ by using excursion theory. Recall that H^x denotes the excursion law for Brownian motion in D_*. We will estimate the H^x-measure of the family F_a of excursions with the property that $|\arg \mathrm{e}(0) - \arg \mathrm{e}(\zeta-)| \le 2\pi$ and $\sup_{t\in[0,\zeta(\mathrm{e}))} |\arg \mathrm{e}(0) - \arg \mathrm{e}(t)| \ge a$, for $a \ge 4\pi$. Note that this quantity does not depend on x. Let $\widehat{H}^x$ be the excursion law for Brownian motion in $D_- = \{z \in \mathbb{C} : \mathrm{Re} z < 0\}$ starting from $x \in \partial D_-$. Excursion laws are conformally invariant, up to a multiplicative constant (see (Burdzy, 1987, Prop. 10.1)). The exponential function $f(z) = \exp(z)$ maps D_- onto $D_* \setminus \{0\}$ and is locally conformal, up to the boundary. Hence, for some constant c_4, $H^x(F_a) = c_4 \widehat{H}^y(\widehat{F}_a)$, where $\widehat{F}_a$ is the family of excursions with the property that $|\mathrm{Im}\, \mathrm{e}(0) - \mathrm{Im}\, \mathrm{e}(\zeta-)| \le 2\pi$ and $\sup_{t\in[0,\zeta(\mathrm{e}))} |\mathrm{Im}\, \mathrm{e}(0) - \mathrm{Im}\, \mathrm{e}(t)| \ge a$. If we normalize all excursion laws as in (2.5) then it is easy to check that $c_4 = 1$ (although our argument does not depend on the value of this constant). Thus, the equality $H^x(F_a) = \widehat{H}^y(\widehat{F}_a)$ holds for all $x \in \partial D_*$ and $y \in \partial D_-$. By (Burdzy, 1987, Thm. 5.1(v)), for some $c_5 < \infty$,

$$\widehat{H}^x \left(\sup_{t\in[0,\zeta(\mathrm{e}))} |\mathrm{Im}\, \mathrm{e}(0) - \mathrm{Im}\, \mathrm{e}(t)| \ge a \right) \le c_5/a. \tag{4.27}$$

It is easy to see that if Brownian motion starts in D_- from a point z with $|\mathrm{Im}\, z| > a$ with $a \ge 4\pi$ then the chance that it will exit D_- through the line segment on the imaginary axis between $-2\pi i$ and $2\pi i$ is bounded above by c_6/a. This, (4.27) and the strong Markov property of $\widehat{H}^x$ applied at the time $\inf\{t \in [0, \zeta(\mathrm{e})) : |\mathrm{Im}\, \mathrm{e}(0) - \mathrm{Im}\, \mathrm{e}(t)| \ge a\}$ imply that

$$H^x(F_a) = \widehat{H}^x \left(\widehat{F}_a \right) \le c_5 c_6/a^2 = c_7/a^2. \tag{4.28}$$

Fix some $\alpha > 0$. By the exit system formula (2.4), the probability that there exists an excursion e_t of X such that $L_t > s$ and e_t belongs to $F_{\alpha L_t}$ is equal to

$$\int_s^\infty H^{X_{\sigma(u)}}(F_{\alpha u}) du \le \int_s^\infty c_7/(\alpha u)^2 du = c_8/(\alpha^2 s).$$

This quantity goes to 0 as $s \to \infty$, so for every fixed $\alpha > 0$, with probability 1, there is $s_\alpha = s_\alpha(\omega) < \infty$ such that there are no excursions $\mathrm{e}_t \in F_{\alpha L_t}$ with $L_t > s_\alpha$.

Fix an arbitrarily small $\alpha > 0$ and suppose that t_1 is so large that $\frac{1}{t} \arg^* X_{T_t} \le \mu_0 + \alpha$ and $L_t/t \le 2$ for all $t > t_1$. If $\frac{1}{u} \arg^* X_u \ge \mu_0 + 5\alpha$ for some $u > t_1$ then $|\arg^* X_u - \arg^* X_{T_u}| \ge 4\alpha u \ge 2\alpha L_u$. This means that an excursion starting at T_u belongs to $F_{2\alpha L_u} = F_{2\alpha L_{T_u}}$. Since there are no such excursions beyond some $s_{2\alpha}$, it follows that $\limsup_{t\to\infty} \frac{1}{t} \arg^* X_{T_t} \le \mu_0 + 5\alpha$, a.s. This holds for all rational $\alpha > 0$ simultaneously, a.s., so $\limsup_{t\to\infty} \frac{1}{t} \arg^* X_{T_t} \le \mu_0$, a.s. The matching lower bound for $\liminf$ can be proved analogously. We conclude that for every $z \in D_*$, $\mathbb{P}_z$-a.s., $\lim_{t\to\infty} \frac{1}{t} \arg^* X_{T_t} = \mu_0$.

The proof of (3.4) will be combined with the proof of Theorem 3.15 (iv) given below.

(iv) Since h is C^2 on $\overline{D}_*$, it follows from (2.18) that $\theta(z)$ is C^2 on $\overline{D}_*$, and hence $\theta(x)$ is C^2 on ∂D_*. Moreover $H = h + i\widetilde{h}$ is $C^{2-\varepsilon}$, by Corollary II.3.3 in Garnett and Marshall (2005). By assumption, h is positive and continuous on ∂D_*. Thus $H(\overline{D}_*)$ is a compact subset of $\{\mathrm{Re}\, z > 0\}$ and so by (2.18), $\sup_x |\theta(x)| < \pi/2$. We can now apply parts (i) and (iii) of the theorem to see that part (iv) holds. $\square$

Proof of Theorem 3.5. Fix a Borel measurable function $\theta : \partial D_* \to [-\pi/2, \pi/2]$. First we need to prove that there exists a sequence of C^2 functions $\theta_k : \partial D_* \to (-\pi/2, \pi/2)$ which converges to θ in weak-* topology. For this, we extend θ harmonically to $\overline{D}_*$ and then we let $\theta_k(e^{it}) = \theta(e^{it}(1-1/k))$. Then θ_k's converge to θ in weak-* topology. See (Hoffman, 1962, page 33).

(i) This was essentially proved in (Burdzy and Marshall, 1993, Thm. 1.1). That theorem was concerned with ORBM in a half-plane while the present result is set in a disc. Theorem 3.5(i) can be proved just like (Burdzy and Marshall, 1993, Thm. 1.1) by repeating the arguments given in Burdzy and Marshall (1993) with some minor adjustments. We omit the proof to save space. The Markov property of X follows from that of X^k and the convergence of finite dimensional distributions. Since for each k, the subprocess of X^k before hitting ∂D_* is Brownian motion in D_* before hitting ∂D_*, the same claim applies to the subprocess of X before hitting ∂D_*.

The transition probabilities are the same for each process $|X^k|$ so the process $|X|$ has the same transition probabilities. It follows that X is conservative.

(ii) This claim was shown in the proof of (Burdzy and Marshall, 1993, Thm. 1.1) although it was not a part of the statement of that theorem. See Step 4 on page 214 of Burdzy and Marshall (1993).

(iii) Suppose that $(h_k, \mu_k) \leftrightarrow \theta_k$ and X^k solves the SDE (3.10) except that the initial distribution for X^k is the stationary distribution $h_k(z)dz$. According to Remark 2.6, the measures $h_k(z)dz$ converge to $h(z)dz$. It is easy to see that part (i) of this theorem implies that X^k's converge weakly to a process X satisfying the SDE (2.1), with the initial distribution $h(z)dz$. For every $t \geq 0$ and $k \geq 1$, the distribution of X_t^k is $h_k(z)dz$. Hence, for every $t \geq 0$, the distribution of X_t is $h(z)dz$. This shows that h is a stationary distribution for X satisfying (2.1).

We next show uniqueness of the stationary distribution. As observed in (2.2), for every reflection angle field θ, the radial part $|X|$ of X is a two-dimensional Bessel process confined to $[0,1]$ by reflection at 1. This easily implies that for any initial distribution of X, the distribution of X_1 has a strictly positive density inside $\mathcal{B}(0,1/2)$. If there were more than one invariant measure, at least two of them (say, Q_1 and Q_2) would be mutually singular by Birkhoff's ergodic theorem Sinaĭ (1994). We have just shown that the Lebesgue measure restricted to $\mathcal{B}(0,1/2)$ (let us call it Q_3) is absolutely continuous with respect to the distribution of X_1, so that in particular, $Q_3 \ll Q_1$ and $Q_3 \ll Q_2$. Since $Q_1 \perp Q_2$ by assumption, there exists a set $A \subset \mathcal{B}(0,1/2)$ such that $Q_1(A) = 0$ and $Q_2(\mathcal{B}(0,1/2) \setminus A) = 0$. Therefore, one must have $Q_3(A) = Q_3(\mathcal{B}(0,1/2) \setminus A) = 0$ which contradicts the fact that $Q_3(\mathcal{B}(0,1/2)) \neq 0$.

(iv) The first claim follows easily from the definitions. The second claim follows from the first claim and part (i) of the theorem.

(v) Since θ_k are smooth, (4.10) holds for X^k's, that is,

$$\begin{aligned} \frac{1}{t}\arg X_t^k - \mu_k &= \frac{1}{t}C_t^k + \frac{1}{t}(C_{L_t^k}^k - C_t^k) + \left(\frac{1}{t}\int_0^t \tan(\theta_k(X_s^k))dL_s^k - \mu_k\right) \\ &\quad + \frac{1}{t}(\arg X_t^k - \arg X_{T_t^k}^k), \end{aligned} \tag{4.29}$$

where the symbols with the superscript or subscript k denote objects analogous to those in (4.10). Since X^k's converge to X weakly, we can assume that all these processes are constructed on a single probability space and $X_t^k \to X_t$, a.s., for every fixed t, as $k \to \infty$. In view of (4.29), we can write

$$\frac{1}{t}\arg X_t - \mu_0 = \left(\frac{1}{t}\arg X_t - \frac{1}{t}\arg X_t^k\right) - (\mu_0 - \mu_k) + \frac{1}{t}(C_t^k - C_0^k) + \frac{1}{t}(C_{L_t^k}^k - C_t^k)$$

$$+ \left(\frac{1}{t}\int_0^t \tan(\theta_k(X_s^k))dL_s^k - \mu_k\right) + \frac{1}{t}(\arg X_t^k - \arg X_{T_t^k}^k) + \frac{1}{t}\arg X_{S^k}^k, \tag{4.30}$$

where $S^k = \inf\{t > 0 : X_t^k \in \partial D_*\}$. The distribution of $\frac{1}{t}(C_t^k - C_0^k)$ is Cauchy for every k and t so it suffices to show that all other terms on the right hand side of (4.30) are small for large t and k.

Fix an arbitrarily small $\varepsilon > 0$. Note that (4.17) and (4.18) do not depend on θ so we can apply them for all θ_k. Hence, we can find t_1 so large that for $t \geq t_1$,

$$\mathbb{P}\left(\left|\frac{1}{t}\left(C_{L_t^k}^k - C_t^k\right) + \frac{1}{t}\left(\arg X_t^k - \arg X_{T_t^k}^k\right)\right| \geq \varepsilon\right) < \varepsilon.$$

We will assume without loss of generality that $X_0^k = z \neq 0$, a.s., for all k. (The case $z = 0$ can be dealt with by applying the Markov property at time $t = 1$.) Then $\arg X_{S^k}^k$ has the same distribution for each $k \geq 1$ and so by taking t_1 larger if needed,

$$\mathbb{P}\left(\left|\frac{1}{t}\arg X_{S^k}^k\right| \geq \varepsilon\right) < \varepsilon, \qquad \text{for all } k \geq 1 \text{ and } t \geq t_1.$$

Recall that $X_t^k \to X_t$, a.s. By Remark 2.6 (vi), $\mu_k \to \mu_0$. Thus, for a fixed t, we can make k so large that

$$\mathbb{P}\left(\left|\frac{1}{t}\arg X_t - \frac{1}{t}\arg X_t^k\right| + |\mu_0 - \mu_k| \geq \varepsilon\right) < \varepsilon.$$

Hence, it will suffice to prove that for a fixed $\varepsilon > 0$, some t_1 and k_1, all $t \geq t_1$, $k \geq k_1$ and $z_k \in \overline{D}_*$,

$$\mathbb{P}_{z_k}\left(\left|\frac{1}{t}\int_0^t \tan(\theta_k(X_s^k))dL_s^k - \mu_k\right| > \varepsilon\right) < \varepsilon. \tag{4.31}$$

If we let $Q_k(dx) = h_k(x)dx$ then by (4.11),

$$\mathbb{E}_{Q_k}\left[\frac{1}{t}\int_0^t \tan(\theta_k(X_s^k))dL_s^k\right] = \mu_k. \tag{4.32}$$

Hence, to finish the proof of part (iv) of the theorem, it will suffice to show that

$$\operatorname{Var}\left(\frac{1}{t}\int_0^t \tan(\theta_k(X_s^k))dL_s^k\right) \leq c_1/t. \tag{4.33}$$

We will split the rest of the proof of this part of the theorem into steps.

Step 1. We will recall some results from (Burdzy and Marshall, 1993, Lemmas 2.2-2.3) but we will change the notation.

We will say that $D \subset \mathbb{C}$ is a monotone domain if D is open, connected and for every $z \in D$ and $b > 0$ we have $z + ib \in D$.

Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half-plane. Suppose that $\theta : \partial\mathbb{H} \to [-\pi/2, \pi/2]$ is a Borel measurable function and suppose θ is not equal almost everywhere either to $\pi/2$ or to $-\pi/2$. Then there exists a univalent analytic mapping g of $\mathbb{H}$ onto a monotone domain $D = g(\mathbb{H})$ such that for almost all $x \in \partial\mathbb{H}$, $g(x)$ and $g'(x)$ exist, $g'(x) \neq 0$ and $\arg g'(x) = \theta(x)$. We choose g so that $\lim_{|z|\to\infty} |g(z)| = \infty$. We construct g as follows. Let $\theta : \mathbb{H} \to \mathbb{R}$ be the bounded harmonic extension of our original function $\theta : \partial\mathbb{H} \to [-\pi/2, \pi/2]$ and let $\widetilde{\theta}$ be the harmonic conjugate of θ such that $\widetilde{\theta}(i) = 0$. Define $g : \mathbb{H} \to \mathbb{C}$ by setting $g(i) = i$ and

$$g'(z) = \exp(i(\theta(z) + i\widetilde{\theta}(z))).$$

Then g is one-to-one on $\mathbb{H}$ because $\operatorname{Re} g'(z) > 0$. (See Burdzy and Marshall (1993)). By abuse of notation, we will use the same symbol θ to denote real functions on both ∂D_* and $\partial\mathbb{H}$. Specifically, for $z \in \partial\mathbb{H}$, we let $\theta(z) = \theta(\exp(iz))$, where $\theta(\exp(iz))$ refers to the function $\theta \in \mathfrak{T}$ introduced in the assumptions of Theorem 3.5. Hence, in this proof, $\theta : \partial\mathbb{H} \to \mathbb{R}$ is a periodic function with period 2π. It follows that g is also periodic with period 2π, up to an additive constant. That is, $g(z + 2\pi) = g(z) + d$ for all $z \in \mathbb{H}$, where $d = g(i + 2\pi) - g(i)$. The constant d is non-zero since $\operatorname{Re} g' > 0$.

Suppose that $\theta_k : \partial D_* \to (-\pi/2, \pi/2)$ are C^2-functions which converge weak-*to θ as $k \to \infty$. Let g_k and $D_k := g_k(\mathbb{H})$ correspond to θ_k in the same way as g and $D = g(\mathbb{H})$ correspond to θ. Note that $g_k(z + 2\pi) = g_k(z) + d_k$ for some constant d_k. Moreover if $\varepsilon > 0$, then $g_k(z + i\varepsilon)$ converges to $g(z + i\varepsilon)$ uniformly in $z \in \mathbb{R}$ and $d_k \to d$. Indeed, by weak-* convergence of $\theta_k \in \mathfrak{T}$, we conclude uniform convergence of $\theta_k(z) + i\widetilde{\theta}(z)$ on the compact set $\{z : |z| = e^{-\varepsilon}\}$, and hence g'_k converges uniformly on $I = \{z : 0 \le \operatorname{Re} z \le 2\pi, \operatorname{Im} z = \varepsilon\}$. Integration then shows that $d_k \to d$ and hence g_k converges uniformly to g on $\mathbb{R} + i\varepsilon$. Let $f(z) = \exp(ig^{-1}(z))$, for $z \in D$ and $f_k(z) = \exp(ig_k^{-1}(z))$ for $z \in D_k$. Then f and f_k are locally conformal maps of D and D_k onto $D_* \setminus \{0\}$ which are periodic with periods d and d_k, respectively.

The monotone domains D_k converge to D in the following sense.

(a) If B is open and such that $B \cap \partial D \neq \emptyset$, there is a $k_0 = k_0(B)$ such that $B \cap \partial D_k \neq \emptyset$ for all $k \ge k_0$.
(b) If B is connected and open, with $B \cap D \neq \emptyset$ and $B \subset D_k$ for infinitely many k, then $B \subset D$.
(c) If K is compact and $K \subset D$ then $K \subset D_k$ for all $k \ge k_0 = k_0(K)$.

We invoke conformal invariance of ORBM as in (4.5)-(4.7). For $x \in \partial D_k$ such that $f_k(x) = z \in \partial D_*$, let $\widehat{\mathbf{v}}_k(x) = i \sec\theta_k(z)$. In other words, $\widehat{\mathbf{v}}_k$ is the conformal (inverse) image of the vector of reflection $\mathbf{v}_{\theta_k}$. Suppose that $\widehat{B}$ is a two-dimensional Brownian motion and consider the Skorokhod SDE

$$\widehat{X}^k_t = \widehat{x}_k + \widehat{B}_t + \int_0^t \widehat{\mathbf{v}}_k(\widehat{X}^k_s) d\widehat{L}^k_s, \tag{4.34}$$

where $\widehat{L}^k$ is the local time of $\widehat{X}^k$ on ∂D_k. The process $\widehat{X}^k$ is reflected Brownian motion in D_k with the oblique angle of reflection θ_k. If $c_k(t) = \int_0^t |f_k'(\widehat{X}_s^k)|^2 ds$ then the process $X_t^k = f_k(\widehat{X}^k_{c_k(t)})$ is reflected Brownian motion in D_* with the oblique angle of reflection θ_k.

Let $K_k = f_k^{-1}(\partial\mathcal{B}(0,1/2))$. Note that K_k is the image under the map g_k of the horizontal line $\{z : \operatorname{Im} z = \ln 2\}$, and so K_k is an analytic curve. Let $a_k = \operatorname{Re} d_k = \operatorname{Re}(g_k(2\pi) - g_k(0))$ and for $z \in \partial D_k$, let $R_k(z) = \{x \in \partial D_k : |\operatorname{Re} x - \operatorname{Re} z| \geq a_k\}$. Let $\widehat{T}^k(A) = \inf\{t \geq 0 : \widehat{X}_t^k \in A\}$. We will show that for every θ there exists $p_1 > 0$ such that for every approximating sequence $\{\theta_k\}$ there exists k_1 such that for any $k \geq k_1$ and $z_k \in \partial D_k$,

$$\mathbb{P}_{z_k}(\widehat{T}^k(K_k) < \widehat{T}^k(R_k)) \geq p_1. \tag{4.35}$$

Let $[x,z]$ denote the line segment between $x, z \in \mathbb{C}$. For every θ there exist $a, b \in (0,\infty)$ such that for every approximating sequence $\{\theta_k\}$ there exists k_1 such that for any $k \geq k_1$ and $z \in \partial D_k$ we have $a_k \geq a$ and $K_k \cap [z, z+ib] \neq \emptyset$.

With probability greater than $p_2 > 0$, Brownian motion starting from 0 will hit the line $\{z : \operatorname{Im} z = 2b\}$ before hitting the lines $\{z : |\operatorname{Re} z| = a/2\}$, and then it will cross the imaginary axis before hitting any of the lines $\{z : |\operatorname{Re} z| = a\}$ or $\{z : \operatorname{Im} z = b\}$. Since $\int_0^t \widehat{\mathbf{v}}_k(\widehat{X}_s^k) d\widehat{L}_s^k$ is a purely imaginary number with non-negative imaginary part, this implies that with probability greater than p_2, the process $\widehat{X}^k$ starting from $z_k \in \partial D_k$ will hit the line $\{z : \operatorname{Im} z - \operatorname{Im} z_k = 2b\}$ before hitting the lines $\{z : |\operatorname{Re} z - \operatorname{Re} z_k| = a/2\}$, and then it will cross the line $\{z : \operatorname{Re} z = \operatorname{Re} z_k\}$ before hitting any of the lines $\{z : |\operatorname{Re} z - \operatorname{Re} z_k| = a\}$ or $\{z : \operatorname{Im} z - \operatorname{Im} z_k = b\}$. If the trajectory of $\widehat{X}^k$ follows a path described above then, in view of the definitions of a and b, it will cross K_k before hitting R_k. We conclude that (4.35) holds with $p_1 = p_2 > 0$.

Let

$$\begin{aligned} T^k(A) &= \inf\{t \geq 0 : X_t^k \in A\}, \\ T_b^k &= T^k(\mathcal{B}(0,1/2)), \\ T_*^k &= \inf\{t \geq 0 : X_t^k \in \partial D_*, |\arg X_t^k - \arg X_0^k| \geq 2\pi\}. \end{aligned}$$

By the conformal invariance of ORBM, (4.35) implies that

$$\mathbb{P}_{z_k}(T_b^k < T_*^k) \geq p_1, \qquad \text{for all } k \text{ and } z_k \in \partial D_*. \tag{4.36}$$

Step 2. We will estimate the variance of $\int_0^1 \tan(\theta_k(X_s^k)) dL_s^k$.

Let $S_1^k = T^k(\partial D_* \cup \partial\mathcal{B}(0,1/2))$. The probability that Brownian motion will make a loop in the annulus $D_* \setminus \mathcal{B}(0,1/2)$ (that is, $\arg X^k$ will increase or decrease by 2π) before hitting the boundary of the annulus is less than $p_3 < 1$. This implies that, for any $z \in \overline{D}_*$,

$$\mathbb{P}_z\left(|\arg X^k_{S_1^k} - \arg X_0^k| \leq 2\pi\right) \geq 1 - p_3. \tag{4.37}$$

This and an easy inductive argument based on the strong Markov property applied at the times when consecutive loops are completed shows that there exists n so large that for any $z \in \overline{D}_*$,

$$\mathbb{P}_z\left(|\arg X^k_{S_1^k} - \arg X_0^k| \geq n2\pi\right) \leq p_1/4, \tag{4.38}$$

where p_1 is as in (4.36). Fix such an n and let

$$\begin{aligned} S_2^k &= \inf\{t \geq 0 : |\arg X_t^k - \arg X_0^k| \geq (n+1)2\pi\}, \\ S_3^k &= \inf\{t \geq S_2^k : |\arg X_t^k - \arg X_{S_k^2}^k| \geq n2\pi\}, \\ S_4^k &= \inf\{t \geq 0 : |\arg X_t^k - \arg X_0^k| \geq (2n+1)2\pi\}, \\ S_{5,j}^k &= \inf\{t \geq 0 : |\arg X_t^k - \arg X_0^k| \geq j(2n+2)2\pi\}. \end{aligned}$$

By (4.36), we have for $z \in \partial D_*$,

$$\mathbb{P}_z(T_b^k \leq T_*^k \wedge S_2^k) + \mathbb{P}_z(S_2^k \leq T_b^k \leq T_*^k) \geq p_1.$$

It follows that either

$$\mathbb{P}_z(T_b^k \leq T_*^k \wedge S_2^k) \geq p_1/2, \tag{4.39}$$

or

$$\mathbb{P}_z(S_2^k \leq T_b^k \leq T_*^k) \geq p_1/2. \tag{4.40}$$

Suppose that the last estimate holds. By (4.38) and the strong Markov property applied at S_2^k,

$$\mathbb{P}_z(S_2^k \leq S_3^k \leq T_b^k \leq T_*^k) \leq p_1/4,$$

so, in view of (4.40),

$$\mathbb{P}_z(S_2^k \leq T_b^k \leq S_3^k \wedge T_*^k) \geq p_1/4.$$

It follows from this and (4.39) that

$$\mathbb{P}_z(T_b^k \leq S_3^k) \geq p_1/4,$$

and, therefore, for $z \in \partial D_*$,

$$\mathbb{P}_z(T_b^k \leq S_4^k) \geq p_1/4.$$

We combine this with (4.37) using the strong Markov property at S_1^k to see that for $z \in \overline{D}_*$,

$$\mathbb{P}_z(T_b^k \leq S_{5,1}^k) \geq (1-p_3)p_1/4 =: p_4 > 0.$$

Applying the strong Markov property repeatedly at $S_{5,j}^k$'s, we see that for $z \in \overline{D}_*$ and $j \geq 1$,

$$\mathbb{P}_z\left(T_b^k \geq S_{5,j}^k\right) \leq (1-p_4)^j.$$

In other words,

$$\mathbb{P}_z\left(|\arg X_{T_b^k}^k - \arg X_0^k| \geq j(2n+2)2\pi\right) \leq (1-p_4)^j. \tag{4.41}$$

Let X^0 be the ORBM corresponding to $\theta \equiv 0$. It is easy to see that

$$\arg X_t^k - \arg X_0^k - \int_0^t \tan(\theta_k(X_s^k))dL_s^k$$

has the same distribution as $\arg X_t^0 - \arg X_0^0$. The estimate (4.41) applies to X^0; to prove that, one can apply the same argument as the one for X^k's or a direct elementary proof. Since

$$\begin{aligned}
&\int_0^t \tan(\theta_k(X_s^k))dL_s^k \\
&= \left(\arg X_t^k - \arg X_0^k\right) - \left(\arg X_t^k - \arg X_0^k - \int_0^t \tan(\theta_k(X_s^k))dL_s^k\right),
\end{aligned}$$

and (4.41) applies to both quantities within parentheses, we obtain for $z \in \overline{D}_*$ and $j \geq 1$,

$$\begin{aligned}
&\mathbb{P}_z\left(\left|\int_0^{T_b^k} \tan(\theta_k(X_s^k))dL_s^k\right| \geq 2j(2n+2)2\pi\right) \\
&\leq \mathbb{P}_z\left(\left|\arg X_{T_b^k}^k - \arg X_0^k\right| \geq j(2n+2)2\pi\right) \\
&\quad + \mathbb{P}_z\left(\left|\arg X_{T_b^k}^k - \arg X_0^k - \int_0^{T_b^k} \tan(\theta_k(X_s^k))dL_s^k\right| \geq j(2n+2)2\pi\right) \\
&\leq 2(1-p_4)^j.
\end{aligned}$$

This implies that for some $c_2 < \infty$ and all $z \in \overline{D}_*$ and all k,

$$\mathbb{E}_z\left[\left|\int_0^{T_b^k} \tan(\theta_k(X_s^k))dL_s^k\right|^3\right] \leq c_2. \tag{4.42}$$

Let $V_0 = U_1 = 0$, and for $m \geq 1$,

$$\begin{aligned}
V_m &= \inf\{t \geq U_m : X_t^k \in \mathcal{B}(0,1/2)\}, \\
U_{m+1} &= \inf\{t \geq V_m : X_t^k \notin \mathcal{B}(0,3/4)\}.
\end{aligned}$$

Since $\mathbb{P}(U_{m+1} - V_m > 1 \mid \mathcal{F}_{V_m}) > p_5 > 0$, we have

$$\mathbb{P}(U_m \leq 1) \leq c_3(1-p_5)^m. \tag{4.43}$$

Note that the local time L^k does not increase on intervals $[V_m, U_{m+1}]$. Hence

$$\int_0^1 \tan(\theta_k(X_s^k))dL_s^k = \sum_{m=1}^{\infty} \int_{U_m \wedge 1}^{V_m \wedge 1} \tan(\theta_k(X_s^k))dL_s^k, \tag{4.44}$$

and, therefore,

$$
\begin{aligned}
&\left|\int_0^1 \tan(\theta_k(X_s^k))dL_s^k\right|^3 = \left|\sum_{m=1}^{\infty}\int_{U_m\wedge 1}^{V_m\wedge 1}\tan(\theta_k(X_s^k))dL_s^k\right|^3\\
&\le 3\sum_{m=1}^{\infty}\sum_{i\le m}\sum_{j\le m}\left|\mathbf{1}_{\{U_m<1\}}\int_{U_m\wedge 1}^{V_m\wedge 1}\tan(\theta_k(X_s^k))dL_s^k\right|\cdot\left|\mathbf{1}_{\{U_i<1\}}\int_{U_i\wedge 1}^{V_i\wedge 1}\tan(\theta_k(X_s^k))dL_s^k\right|\\
&\qquad\times\left|\mathbf{1}_{\{U_j<1\}}\int_{U_j\wedge 1}^{V_j\wedge 1}\tan(\theta_k(X_s^k))dL_s^k\right|\\
&\le 3\sum_{m=1}^{\infty}\sum_{i\le m}\sum_{j\le m}\left[\mathbf{1}_{\{U_m<1\}}\left|\int_{U_m\wedge 1}^{V_m\wedge 1}\tan(\theta_k(X_s^k))dL_s^k\right|^3+\mathbf{1}_{\{U_i<1\}}\left|\int_{U_i\wedge 1}^{V_i\wedge 1}\tan(\theta_k(X_s^k))dL_s^k\right|^3\right.\\
&\qquad\left.+\mathbf{1}_{\{U_j<1\}}\left|\int_{U_j\wedge 1}^{V_j\wedge 1}\tan(\theta_k(X_s^k))dL_s^k\right|^3\right].
\end{aligned}
$$

This, (4.42) and (4.43) imply that for some $c_4<\infty$, all $z\in\overline{D}_*$ and all k,

$$
\mathbb{E}_z\left[\left|\int_0^1 \tan(\theta_k(X_s^k))dL_s^k\right|^3\right]\le 3\sum_{m=1}^{\infty}\sum_{i\le m}\sum_{j\le m}3c_3(1-p_5)^m c_2<c_4. \tag{4.45}
$$

Step 3. For a fixed $z\in\overline{D}_*$ and all k, the processes $\{|X_t^k|, t\ge 0\}$ have the same distribution, that of 2-dimensional Bessel process on $[0,1]$, reflected at 1. Hence, $\mathbb{P}_z\left(|X_{1/2}^k|<1/4\right)>p_6$, where p_6 does not depend on $z\in\overline{D}_*$ and k. This and the Markov property at time 1/2 can be used to show that the density of the distribution of X_1^k under $\mathbb{P}_z$ is greater than $c_5>0$ on $\mathcal{B}(0,1/2)$, where c_5 does not depend on $z\in\overline{D}_*$ and k.

Let $\mathbb{P}_x^k$ denote the distribution of the process X^k starting from x. Consider $z\in\overline{D}_*$. We will construct a process X^k with distribution $\mathbb{P}_z^k$ in a special way. First we will construct i.i.d. random vectors $A_1,A_2,A_3,\dots$ The distribution of each A_j is partly continuous, with density c_5 in $\mathcal{B}(0,1/2)$. With probability $1-c_5\pi/4$, A_j takes value $\mathbf{\Delta}$ (the cemetery state). Let q_1^k be the density of X_1^k under the distributions $\mathbb{P}_z^k$. Let B_1 be a random vector with density $q_1^k(x)-c_5\mathbf{1}_{\mathcal{B}(0,1/2)}(x)$ on D_*. With probability $c_5\pi/4$, B_1 takes value $\mathbf{\Delta}$. We construct B_1 so that it is equal to $\mathbf{\Delta}$ if and only if $A_1\ne\mathbf{\Delta}$. Moreover, we make the conditional distribution of B_1 given $\{B_1\ne\mathbf{\Delta}\}$ independent of A_j's.

In the following construction, the expression "Markov bridge" will refer to the Markov bridge corresponding to $\mathbb{P}^k$. If $A_1\in\mathcal{B}(0,1/2)$ then we let $\{X_t^k, 0\le t\le 1\}$ be the Markov bridge between the points in time-space $(0,z)$ and $(1,A_1)$. If $A_1=\mathbf{\Delta}$ then we let $\{X_t^k, 0\le t\le 1\}$ be the Markov bridge between the points $(0,z)$ and $(1,B_1)$, otherwise independent of A_j's and B_1.

We continue by induction. Suppose that $\{X_t^k, 0\le t\le n\}$ has been defined. Let $q_{n+1}^k(X_n^k,x)$ be the density of X_1^k under the distribution $\mathbb{P}_{X_n^k}^k$. Let B_{n+1} be a random vector with density $q_{n+1}^k(X_n^k,x)-c_5\mathbf{1}_{\mathcal{B}(0,1/2)}(x)$ on D_*. With probability $c_5\pi/4$, this random vector takes value $\mathbf{\Delta}$. We construct B_{n+1} so that it is equal to $\mathbf{\Delta}$ if and only if $A_{n+1}\ne\mathbf{\Delta}$. Moreover, we make the conditional distribution of B_{n+1} given $\{B_{n+1} \neq \mathbf{\Delta}\}$ independent of A_j's and $\{X^k_t, 0 \le t \le n\}$, except that it has the density $q^k_{n+1}(X^k_n, x) - c_5 \mathbf{1}_{\mathcal{B}(0,1/2)}(x)$ on D_*.

If $A_{n+1} \in \mathcal{B}(0, 1/2)$ then we let $\{X^k_t, n \le t \le n+1\}$ be the Markov bridge between the points in time-space (n, X^k_n) and $(n+1, A_{n+1})$, otherwise independent of A_j's and $\{X^k_t, 0 \le t \le n\}$. If $A_{n+1} = \mathbf{\Delta}$ then we let $\{X^k_t, n \le t \le n+1\}$ be the Markov bridge between (n, X^k_n) and $(n+1, B_{n+1})$, otherwise independent of A_j's and $\{X^k_t, 0 \le t \le n\}$. It is easy to check that this inductive construction yields a process $\{X^k_t, t \ge 0\}$ with distribution $\mathbb{P}^k_z$.

Let $\Gamma^k_n = \int_n^{n+1} \tan(\theta_k(X^k_s)) dL^k_s$. Let $\mathbf{A}_n = \bigcup_{j=1}^n \{A_n \neq \mathbf{\Delta}\}$ and note that $\mathbb{P}(\mathbf{A}^c_n) = (1 - c_5\pi/4)^n =: c_6^n$, where $c_6 < 1$. If $\mathbf{A}_n$ holds then the trajectory of $\{X^k_t, n \le t \le n+1\}$ does not depend on X^k_1. Hence, $\operatorname{Cov}(\Gamma^k_1, \Gamma^k_n \mathbf{1}_{\mathbf{A}_n}) = 0$. We have,

$$\begin{aligned}
\operatorname{Cov}(\Gamma^k_1, \Gamma^k_n) &= \operatorname{Cov}(\Gamma^k_1, \Gamma^k_n \mathbf{1}_{\mathbf{A}_n} + \Gamma^k_n \mathbf{1}_{\mathbf{A}^c_n}) = \operatorname{Cov}(\Gamma^k_1, \Gamma^k_n \mathbf{1}_{\mathbf{A}_n}) + \operatorname{Cov}(\Gamma^k_1, \Gamma^k_n \mathbf{1}_{\mathbf{A}^c_n}) \\
&= \operatorname{Cov}(\Gamma^k_1, \Gamma^k_n \mathbf{1}_{\mathbf{A}^c_n}) = \mathbb{E}(\Gamma^k_1 \Gamma^k_n \mathbf{1}_{\mathbf{A}^c_n}) - \mathbb{E}\,\Gamma^k_1\, \mathbb{E}(\Gamma^k_n \mathbf{1}_{\mathbf{A}^c_n}),
\end{aligned}$$

so, in view of (4.45), for some $c_{10} < 1$,

$$\begin{aligned}
|\operatorname{Cov}(\Gamma^k_1, \Gamma^k_n)| &\le (\mathbb{E}\,|\Gamma^k_1|^3)^{1/3} (\mathbb{E}\,|\Gamma^k_n|^3)^{1/3} (\mathbb{E}\,\mathbf{1}^3_{\mathbf{A}^c_n})^{1/3} + \mathbb{E}\,\Gamma^k_1 (\mathbb{E}(\Gamma^k_n)^2)^{1/2} (\mathbb{E}\,\mathbf{1}^2_{\mathbf{A}^c_n})^{1/2} \\
&\le c_7 c_4^{n/3} + c_8 c_4^{n/2} \le c_9 c_{10}^n.
\end{aligned}$$

It is easy to see that the estimate applies also to $n = 1$ (possibly with new values of the constants). This implies that

$$\begin{aligned}
&\operatorname{Var}\left(\int_0^n \tan(\theta_k(X^k_s)) dL^k_s \right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \operatorname{Cov}\left(\int_{i-1}^i \tan(\theta_k(X^k_s)) dL^k_s, \int_{j-1}^j \tan(\theta_k(X^k_s)) dL^k_s \right) \\
&\le \sum_{i=1}^n \sum_{j=1}^n c_9 c_{10}^{|i-j|} \le c_{11} n.
\end{aligned}$$

It is elementary to check that the estimate also applies with non-integer upper limit, that is, for any $t > 1$,

$$\operatorname{Var}\left(\int_0^t \tan(\theta_k(X^k_s)) dL^k_s \right) \le c_{11} t.$$

This completes the proof of (4.33) and hence the proof of part (v) of the theorem.

(vi) The claim follows from the ergodic theorem if we show that under the stationary distribution $h(x)dx$,

$$\mathbb{E}_h \left[\mathbf{arg}^* X_1 \right] = \mu_0. \tag{4.46}$$

Recall that

$$\lim_{k \to \infty} \mu_k = \mu_0. \tag{4.47}$$

Theorem 3.2 (iii) implies that

$$\mathbb{E}_{h_k}\left[\arg^* X_1^k\right] = \mu_k. \tag{4.48}$$

It follows easily from definitions of $\arg^*$ and $\mathbf{arg}^*$, and Theorem 3.5(iv) that $\arg^* X_1^k \to \mathbf{arg}^* X_1$ in distribution. Hence, in view of (4.47)-(4.48), the proof of (4.46) will be complete if we prove that the family $\{\arg^* X_1^k\}_{k\geq 1}$ is uniformly integrable.

The following formula can be derived in the same way as (4.10) has been derived,

$$\arg^* X_1^k = C^*_{L_1^k} + \int_0^1 \tan(\theta_k(X_s^k))dL_s^k + \left(\arg X_1^k - \arg X^k_{T_1^k}\right). \tag{4.49}$$

Here C^* is a Cauchy process with jumps larger than 2π removed.

Recall that $S^k = \inf\{t > 0 : X_t^k \in \partial D_*\}$ and $T_1^k = \inf\{t > 1 : X_t^k \in \partial D_*\}$. So by the Markov property of X^k, under the stationary measure $h_k(x)dx$, $\{X_s^k - X_1^k; 1 \leq s \leq T_1^k\}$ has the same distribution as that of $\{X_s^k - X_0^k; 0 \leq s \leq S^k\}$. It follows from the paragraph following (2.2) that under the stationary measure $h_k(x)dx$, $Y^k = |X^k|$ is a stationary 2-dimensional Bessel process in $(0, 1]$ reflected at 1. Let $\sigma_k(a, b] = \int_{\{a<|x|\leq b\}} h_k(x)dx$. Then $\sigma_k(dr)$ is the stationary probability distribution of Y^k so it is independent of k. This and the rotational invariance of Brownian motion imply that the distribution of $\arg X_0^k - \arg X^k_{S^k}$ does not depend on k. By an earlier remark, the distribution of $\arg X_1^k - \arg X^k_{T_1^k}$ is the same so it does not depend on k either. Hence, the family $\left\{\arg X_1^k - \arg X^k_{T_1^k}\right\}$, $k \geq 1$, is uniformly integrable. The distribution of L_1^k does not depend on k so the same applies to $C^*_{L_1^k}$. Random variables $\int_0^1 \tan(\theta_k(X_s^k))dL_s^k$ are uniformly integrable by (4.45). All these remarks taken together with (4.49) show that the family $\{\arg^* X_1^k\}_{k\geq 1}$ is uniformly integrable. This completes the proof of part (vi) of the theorem.

(vii) An explicit integral test was given in Burdzy and Marshall (1992): The ORBM in $D_+ = \{z : \operatorname{Im} z > 0\}$ with angle of reflection θ hits 0 with positive probability if and only if

$$\int_0^1 \frac{1}{y}\operatorname{Re}\,\exp\left(i\left(\theta(iy) + i\widetilde{\theta}(iy)\right)\right)dy < \infty, \tag{4.50}$$

where $\theta(z)$ is the bounded harmonic extension of θ to D_+ and $\widetilde{\theta}$ is the harmonic conjugate of θ vanishing at $z = i$. In Burdzy and Marshall (1992) there was the added assumption that $\theta \in C^{1+\varepsilon}$, for some $\varepsilon > 0$, except possibly at 0. As noted in Burdzy and Marshall (1993), the same result holds if we only assume θ is measurable and $|\theta| \leq \pi/2$. One way to transfer this result to $\theta \in \mathcal{T}$ is to set $\theta_1(t) = \theta(e^{it})$, for $t \in \mathbb{R}$, and $\theta_1(z) = \theta(e^{iz})$ for $z \in D_+$ as before. Then

$$\int_0^1 \frac{1}{y}\operatorname{Re}\,\exp\left(i\left(\theta_1(iy) + i\widetilde{\theta}_1(iy)\right)\right)dy = \int_0^1 \frac{1}{y}\operatorname{Re}\,\exp\left(i(\theta + i\widetilde{\theta})(e^{-y})\right)dy.$$

Setting $r = e^{-y}$, we have $y = \ln 1/r \sim 1 - r$ on $[e^{-1}, 1]$ and so ORBM hits 1 with positive probability in D_* if and only if the left-hand side of (3.13) is finite for $x = 1$. By (2.19),

$$1/(h + i\widetilde{h} - i\mu_0/\pi) = \pi\cos\theta(0)e^{i(\theta+i\widetilde{\theta})}$$

and by taking real parts, the two integrals in (3.13) are equal.

Suppose that for some $z_0 \in D_*$ and $x \in \partial D_*$, $\mathbb{P}_{z_0}(x \in \Gamma^\theta_X) > 0$. A simple coupling argument shows that for some $r > 0$ and $p > 0$, $\mathbb{P}_z(x \in \Gamma^\theta_{X[0,1]}) \ge p$ for all $z \in \mathcal{B}(z_0, r) \subset D_*$. Since for every $k \ge 1$, X_t returns to $\mathcal{B}(z_0, r)$ for some $t \ge k$ with probability one, we have by this "renewal property" that $\mathbb{P}_z(x \in \Gamma^\theta_X) = 1$ for all $z \in \mathcal{B}(z_0, r)$.

(viii) Let ρ denote the Prokhorov distance between probability measures ((Billingsley, 1999, App. III)). For any stochastic processes V and Z, we will write $\rho(V, Z)$ to denote the distance between their distributions relative to M_1 distance between trajectories. For every k, one can find a sequence $(\theta^n_k)_{n\ge1}$ of C^2 functions with values in $(-\pi/2, \pi/2)$ which converges to $\bar\theta_k$ as $n \to \infty$ in weak-* topology. Recall that $\bar X_k$ are defined relative to $\bar\theta_k$ in the same way that X is defined relative to θ. Processes X^k are defined by (3.10) relative to θ_k. By part (i) of the theorem, one can find a sequence $\theta^{n_k}_k : \partial D_* \to (-\pi/2, \pi/2)$ with the following properties. Let X^{k,n_k} be the solution to (3.10) relative to $\theta^{n_k}_k$. Then $\rho(X^{k,n_k}, \bar X^k) < 1/k$. Moreover, we can choose n_k's so large that the sequence $(\theta^{n_k}_k)_{k\ge1}$ converges to θ in weak-* topology. Since the sequence $(\theta_1, \theta^{n_1}_1, \theta_2, \theta^{n_2}_2, \theta_3, \theta^{n_3}_3, \dots)$ converges to θ, the sequence of processes $X^1, X^{1,n_1}, X^2, X^{2,n_2}, X^3, X^{3,n_3}, \dots$ converges in distribution to a process X', by part (i) of the theorem. We must have $X = X'$ in distribution, because $(\theta_k)_{k\ge1}$ is a subsequence of $(\theta_1, \theta^{n_1}_1, \theta_2, \theta^{n_2}_2, \theta_3, \theta^{n_3}_3, \dots)$. We see that $\rho(X^{k,n_k}, X) \to 0$. Since $\rho(X^{k,n_k}, \bar X^k) < 1/k$, we obtain $\rho(\bar X^k, X) \to 0$ as $k \to \infty$. □

Proof of Proposition 3.7. The integral in (3.13) is equal to

$$\int_0^1 \frac{1}{1-r} \int_{\partial D_*} \frac{1-r^2}{|z-rx|^2} d\sigma(z) dr = \int_{\partial D_*} \int_0^1 \frac{1+r}{|1-rx\bar z|^2} dr d\sigma(z).$$

Let $w = x\bar z$. Then $|w| = 1$ and

$$\frac{1}{|1-rw|^2} = \frac{1}{(1-rw)(1-r\overline{w})} = \frac{1}{\overline{w}-w}\left(\frac{-w}{1-rw} - \frac{\overline{w}}{1-r\overline{w}}\right).$$

So

$$\int_0^1 \frac{1}{|1-rw|^2} dr = \frac{1}{\overline{w}-w} \ln\frac{1-w}{1-\overline{w}} = \frac{\arg(1-w)}{|1-w| \sin\arg(1-w)} \sim \frac{1}{|1-w|}.$$

Thus the integral in (3.13) is finite if and only if (3.17) holds. □

Proof of Proposition 3.11. An application of the Riemann mapping theorem shows that it suffices to prove the proposition for $D = D_*$.

(i) The expected occupation measure for an excursion law H^x is a constant multiple of $K_x(\,\cdot\,)$ by (2.8). According to the definition, the ERBM is a "mixture" of excursion laws. This easily implies that the stationary distribution for X has the density that is proportional to $\int_{\partial D_*} K_x(y)\nu(dx)$.

(iii) The function h has a representation $h(y) = \int_{\partial D_*} K_x(y)\nu(dx)$. If one constructs an ERBM corresponding to ν then the stationary measure of this process is h by part (i) of the proposition. □

Proof of Theorem 3.12. (i) Since $\lim_{k\to\infty} \mathrm{dist}(x_k, \partial D_*) = 0$, every subsequence of x_k contains a further subsequence that converges to some point in ∂D_*. We will assume that the whole sequence x_k converges to a point $x_\infty \in \partial D_*$. We will show that the limit distribution of X^k does not depend on x_∞. Hence, the result holds for every sequence satisfying $\lim_{k\to\infty} \mathrm{dist}(x_k, \partial D_*) = 0$.

As was noted in the paragraph following (2.2), for any $r_0 \in [0,1]$, $t \geq 0$ and $\theta_1, \theta_2 \in \mathcal{T}$, if X^k is an ORBM in D_* with the angle of reflection θ_k and $|X^k_0| = r_0$ for $k = 1,2$, then the distributions of $|X^1_t|$ and $|X^2_t|$ are identical. Suppose that X is an ORBM. Then $\mathbb{P}(|X_t| \in [1-\varepsilon, 1]) \leq c\varepsilon$ for some c and all $\varepsilon \geq 0$. Fix an arbitrary $\varepsilon \in (0,1)$. Let $\mathcal{E}^*_\varepsilon = \{\mathrm{e}^1, \mathrm{e}^2, \dots\}$ be the set of all excursions of X from ∂D_* which enter the ball $\mathcal{B}(0, 1-\varepsilon)$, ordered according to their starting times. Let $S_n = S_n(\varepsilon) = \inf\{t \geq 0 : \mathrm{e}^n_t \in \mathcal{B}(0, 1-\varepsilon)\}$. It follows from the rotation invariance of Brownian motion that the distribution of $\{\exp(-i \arg \mathrm{e}^n_{S_n})\mathrm{e}^n_t, t \geq S_n\}$ (the excursion rotated about 0 so that $\mathrm{e}^n_{S_n}$ is mapped to $1-\varepsilon \in \mathbb{R}$) does not depend on n, θ or the value taken by S_n.

Since the process $\{\mathrm{e}^n_t, t \geq S_n\}$ is Brownian motion killed upon hitting ∂D_*, its trajectory has modulus of continuity $c(\omega)\sqrt{2r|\log r|}$, where $c(\omega)$ is finite for almost all ω (see (Karatzas and Shreve, 1991, Thm. 2.9.25)). If we time-reverse e^n and rotate it so that it starts from 0, then it will have the distribution H^0 conditioned by $\{\exists t > 0 : \mathrm{e}_t \in \mathcal{B}(0, 1-\varepsilon)\}$. Hence, the claim about the modulus continuity can be extended as follows. The modulus of continuity of $\{\mathrm{e}^n_t, t \in (0,\zeta)\}$ is $c_1(\omega)\sqrt{2r|\log r|}$, where $c_1(\omega)$ is finite for almost all ω. This easily implies that for any sequence of random variables V_k which converges to 0 in distribution, processes $\{\exp(-iV_k)\mathrm{e}^n_t, t \geq 0\}$ converge to $\{\mathrm{e}^n_t, t \geq 0\}$ in distribution in the Skorokhod topology as $k \to \infty$. Note that no assumptions on the joint distribution of V_k and $\{\mathrm{e}^n_t, t \geq 0\}$ are needed.

Recall that $h_k(0) = 1/\pi$ for any $(h_k, \mu_{0,k}) \in \mathcal{H}$. Hence $\int_{\partial D_*} h_k(x)dx = 2$ and, therefore, $\nu(\partial D_*) = 2$. It follows that $\nu/2$ is a probability distribution on ∂D_*.

Let $\mathcal{E}^k_\varepsilon$ be defined relative to X^k in the same way as $\mathcal{E}^*_\varepsilon$ has been defined relative to a generic X. We will suppress both ε and k in the notation for excursions, i.e., we will write $\mathcal{E}^k_\varepsilon = \{\mathrm{e}^1, \mathrm{e}^2, \dots\}$. In view of the opening remarks of this proof, it is routine to show that in order to prove part (i) of the theorem, it is sufficient to show that for any fixed $\varepsilon \in (0,1)$ and n, the joint distribution of $(\mathrm{e}^1_0, \mathrm{e}^2_0, \dots, \mathrm{e}^n_0)$ converges to that of a sequence of n i.i.d. random variables with distribution $\nu/2$, as $k \to \infty$.

Let $\sigma^k_t = \inf\{s \geq 0 : L^k_s > t\}$ and $A^k_t = \arg X^k_{\sigma^k_t}$, with the convention that $\arg X^k_{\sigma^k_t} \in [0, 2\pi)$. By abuse of notation, we define θ_k for real x by $\theta_k(x) = \theta_k(e^{ix})$. Let B be Brownian motion in $\mathbb{C}$ starting at the origin and $S^k = \inf\{t > 0 : x_k + B_t \in \partial D_*\}$. Let $\widehat{A}^k_0 = \arg(x_k + B_{S^k})$. Since $x_k \to x_\infty \in \partial D_*$, $a_0 := \lim_{k\to\infty} \widehat{A}^k_0 = \arg x_\infty$ a.s. Let C_t be a Cauchy process with $C_0 = 0$ that is independent of B, and let $\widehat{A}^k_t$ be the solution to the SDE

$$\widehat{A}^k_t = \widehat{A}^k_0 + C_t + \int_0^t \tan\theta_k(\widehat{A}^k_s)ds. \tag{4.51}$$

Clearly, $\widehat{A}^k_0$ has the same distribution as A^k_0. Let $\bar{A}^k_t \in [0, 2\pi)$ be the unique number such that $\bar{A}^k_t = \widehat{A}^k_t + j2\pi$ for some integer j. Then, by the conformal invariance of ORBM's presented in (4.5)-(4.7), the distribution of $\{\bar{A}^k_t, t \geq 0\}$ is the same as that of $\{A^k_t, t \geq 0\}$.

To incorporate our assumptions on h_k and $1/h_k$, we first note that by (2.15) and (2.11)

$$\tan\theta_k(z) = \frac{\mu_k(z)}{\pi h_k(z)} = \frac{\mu_{0,k}}{\pi h_k(z)} - \frac{\widetilde{h}_k(z)}{h_k(z)}, \tag{4.52}$$

for $z \in D_*$. If f is Lipschitz with constant λ, then its modulus of continuity satisfies $\omega_f(\delta) \le \lambda\delta$. By (Garnett, 2007, Thm. III.1.3) the modulus of continuity of $\widetilde{f}$ satisfies

$$\omega_{\widetilde{f}}(\delta) \le C\lambda\delta(1 + \log \pi/\delta),$$

where C is a constant not depending on f or δ. So by assumption (c), $\widetilde{h}_k$ are Dini continuous on $\overline{D}_*$, with constants depending only on λ, not k. We also conclude that each θ_k and $\widetilde{\theta}_k$ are Dini continuous on D_*, and therefore on $\overline{D}_*$, by (2.18). In particular, (4.52) holds for $x \in \partial D_*$.

By a change of variables,

$$\begin{aligned} \widehat{A}^k_{t/\mu_{0,k}} &= \widehat{A}^k_0 + C_{t/\mu_{0,k}} + \int_0^t \tan\theta_k(\widehat{A}^k_{r/\mu_{0,k}}) \frac{dr}{\mu_{0,k}} \\ &= \widehat{A}^k_0 + C_{t/\mu_{0,k}} - \frac{1}{\mu_{0,k}} \int_0^t \frac{\widetilde{h}_k(A^k_{r/\mu_{0,k}})}{h_k(A^k_{r/\mu_{0,k}})} dr + \frac{1}{\pi}\int_0^t \frac{1}{h_k(A^k_{r/\mu_{0,k}})} dr. \end{aligned} \tag{4.53}$$

By assumption (d), $h_k(z) = \int K_z(x) h_k(x)|dx|$ converges to $\int K_z \nu(dx) :\equiv h(z)$, where K_z is the Poisson kernel for $z \in D_*$. Since each h_k is Lipschitz with constant λ on ∂D_* and therefore on $\overline{D}_*$, we have that $|h(z) - h(w)| \le \lambda|z - w|$ for $z, w \in D_*$. Thus h extends to be Lipschitz with constant λ on $\overline{D}_*$ and so $\nu(dx) = h(x)|dx|$.

Recall from Remark 3.13 (iii) that the assumption (c) implies that all functions $1/h_k$ are Lipschitz with the same constant. Without loss of generality, we will assume that the Lipshitz constant for $1/h_k$ is λ. It follows that $1/h$ is Lipshitz with constant λ.

Recall that $a_0 := \lim_{k\to\infty} \widehat{A}^k_0 = \arg x_\infty$. By abuse of notation, let $h(x) = h(e^{ix})$ for real x and let a_t be the solution to

$$a_t = a_0 + \int_0^t \frac{1}{\pi h(a_s)} ds. \tag{4.54}$$

Let t_1 be such that $a_{t_1} = a_0 + 2\pi$. Since

$$\frac{\partial}{\partial t}\nu([a_0, a_t])/2 = (1/2)\frac{\partial}{\partial t}\int_{a_0}^{a_t} h(b)db = (1/2)\frac{h(a_t)}{\pi h(a_t)} = \frac{1}{2\pi}$$

and $\nu([a_0, a_{t_1}])/2 = \nu([a_0, a_0 + 2\pi])/2 = 1$, we must have $t_1 = 2\pi$. Hence, for $0 \le s \le t \le 2\pi$,

$$\nu([a_s, a_t])/2 = \frac{t - s}{2\pi}. \tag{4.55}$$

It follows from (4.53)-(4.54) that

$$\widehat{A}^k_{t/\mu_{0,k}} - a_t = F^k_t + \frac{1}{\pi}\int_0^t \left(\frac{1}{h(A^k_{r/\mu_{0,k}})} - \frac{1}{h(a_r)} \right) dr,$$

where

$$F^k_t = \widehat{A}^k_0 - a_0 + C_{t/\mu_{0,k}} - \frac{1}{\mu_{0,k}}\int_0^t \frac{\widetilde{h}_k(A^k_{r/\mu_{0,k}})}{h_k(A^k_{r/\mu_{0,k}})} dr + \frac{1}{\pi}\int_0^t \left(\frac{1}{h_k(A^k_{r/\mu_{0,k}})} - \frac{1}{h(A^k_{r/\mu_{0,k}})} \right) dr. \tag{4.56}$$

Since $1/h$ is Lipschitz with constant λ,

$$|\widehat{A}^k_{t/\mu_{0,k}} - a_t| \le \sup_{0\le s\le 2\pi} |F^k_s| + \frac{\lambda}{\pi}\int_0^t \left|\widehat{A}^k_{r/\mu_{0,k}} - a_r\right| dr,$$

for $0 \le t \le 2\pi$. By Grönwall's inequality (see Bellman (1943)),

$$\left|\widehat{A}_{t/\mu_{0,k}} - a_t\right| \le \left(\sup_{0\le s\le 2\pi} |F^k_s|\right) e^{\lambda t/\pi}. \tag{4.57}$$

We claim that

$$\lim_{k\to\infty} \sup_{0\le s\le 2\pi} |F^k_s| = 0, \tag{4.58}$$

in probability. By the definition of a_0, $\lim_k \widehat{A}^k_0 - a_0 = 0$. By assumption (a), $\theta_k(0) = \int \theta_k(x)|dx|/2\pi$ converges to $\pi/2$. But then $\mu_{0,k} = \tan\theta_k(0)$ converges to $+\infty$. Thus $\sup_{0\le t\le 2\pi} C_{t/\mu_{0,k}} = 0$, a.s. Since $\widetilde{h}_k$ and $1/h_k$ are Dini continuous on D_* with constant depending only on λ, and $h_k(0) = 1/\pi$ and $\widetilde{h}_k(0) = 0$, we have that $\widetilde{h}_k/h_k$ is bounded on ∂D_* by a constant independent of k. Thus the first integral in (4.56) also tends to 0.

If $\beta_n(f)$ denotes the n^{th} Cesaro mean of f on ∂D_* then for continuous f, $\beta_n(f)$ converges uniformly on ∂D_* to f, with the difference $\|\beta_n(f) - f\|_\infty$ depending only on the modulus of continuity of f and n. See (Hoffman, 1962, page 18). Since $1/h_k$ and $1/h$ are Lipschitz with constant λ, given $\varepsilon > 0$ we can choose n so that

$$\|1/h_k - \beta_n(1/h_k)\|_\infty < \varepsilon \quad\text{and}\quad \|1/h - \beta_n(1/h)\|_\infty < \varepsilon. \tag{4.59}$$

By assumption (d) h_k converges to h, uniformly on compact subsets of D_*, and since $1/h_k$ are uniformly bounded, $1/h_k$ converges to $1/h$ uniformly on compact subsets of D_*. Since $1/h_k$ are uniformly bounded, this also implies $1/h_k$ converges to $1/h$ weak-* and therefore for k sufficiently large, and n fixed,

$$\|\beta_n(1/h_k) - \beta_n(1/h)\|_\infty < \varepsilon. \tag{4.60}$$

By (4.59), (4.60), and the triangle inequality, $1/h_k$ converges uniformly to $1/h$. We conclude that the second integral in (4.56) tends to 0 as well, proving the claim.

We will need a generalization of the above results (4.57) and (4.58). Let $D_u = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half-plane. Let H^x be the excursion law for Brownian motion in D_*, for excursions starting from $x \in \partial D_*$ and let $\widehat{H}^x$ be the excursion law for Brownian motion in D_u, for excursions starting from $x \in \partial D_u$. The measure $\widehat{H}^0(\mathrm{e}(\zeta-) \in dx)$ is the distribution of the end point of the excursion under $\widehat{H}^0$. It is also the Lévy measure for the Cauchy process. Let

$$\mu_\varepsilon(dx) = \widehat{H}^0\left(\sup_{t\in[0,\zeta)} \operatorname{Im}\mathrm{e}_t < |\log(1-\varepsilon)|, \mathrm{e}(\zeta-) \in dx\right).$$

The measure μ_ε is the Lévy measure for a pure jump process, say C^ε_t, similar to the Cauchy process, except that it has fewer big jumps. We can choose a right continuous version of C^ε, and so $\sup_{0\le s\le t}|C^\varepsilon_s|\to 0$, a.s., as $t\to 0$. We let $\widehat{A}^{k,\varepsilon}_t$ be the solution to the equation analogous to (4.51),

$$\widehat{A}^{k,\varepsilon}_t=\widehat{A}^{k,\varepsilon}_0+C^\varepsilon_t+\int_0^t\tan\theta_k(\widehat{A}^{k,\varepsilon}_s)ds. \tag{4.61}$$

An argument analogous to that showing (4.57) and (4.58) proves that for every fixed $\varepsilon>0$,

$$\sup_{0\le s\le 2\pi}\left|\widehat{A}^{k,\varepsilon}_{s/\mu_{0,k}}-a_s\right|\to 0, \tag{4.62}$$

in probability, as $k\to\infty$.

Recall the definition of $\mathcal{E}^k_\varepsilon=\{\mathrm{e}^1,\mathrm{e}^2,\dots\}$ from the beginning of the proof. We claim that for any fixed $\varepsilon\in(0,1)$ and n, the joint distribution of $(\mathrm{e}^1_0,\mathrm{e}^2_0,\dots,\mathrm{e}^n_0)$ converges to that of a sequence of n i.i.d. random variables with distribution $\nu/2$, as $k\to\infty$.

We will present a special construction of $(\mathrm{e}^1_0,\mathrm{e}^2_0,\dots,\mathrm{e}^n_0)$. The heuristic meaning of the construction is the following. Excursions that reach $\mathcal{B}(0,1-\varepsilon)$ occur as a Poisson process with constant intensity on the local time scale. If we have already observed $\mathrm{e}^1,\mathrm{e}^2,\dots,\mathrm{e}^m$, the next excursion will occur after an exponential waiting time on the local time scale, where the local time has the same distribution as the process $\widehat{A}^{k,\varepsilon}_t$. This process, suitably rescaled, behaves like the function a_t according to (4.62). By (4.55), a point on the boundary chosen in a uniform manner on the a_t scale has the distribution $\nu/2$. We will also need a fact that, on small time intervals, exponential density is almost constant. The process $\widehat{A}^{k,\varepsilon}_t$ represents rapid rotation along the unit circle and the exponential clock will chose a point on the circle according to the distribution very close to $\nu/2$, because the almost constant exponential density (on small intervals) is transformed into the density of $\nu/2$ by the function a_t.

Suppose that excursions $\mathrm{e}^1,\mathrm{e}^2,\dots,\mathrm{e}^m$ have been already generated, for some $m\ge 0$. If $m\ge 1$, let T_m be the time when e^m ended. If $m=0$ then we take T_0 to be the first hitting time of ∂D_* by X^k. Unless stated otherwise, every new random object introduced below will be assumed to be independent from all random objects constructed so far.

By conformal invariance of excursion laws,

$$H^x\left(\exists t\in[0,\zeta):\mathrm{e}_t\in\mathcal{B}(0,1-\varepsilon)\right)=\widehat{H}^0\left(\exists t\in[0,\zeta):\operatorname{Im}\mathrm{e}_t\ge|\log(1-\varepsilon)|\right),$$

and the last quantity is equal to $1/|\log(1-\varepsilon)|$ (see Burdzy (1987) for the justification of both claims).

Consider an exponential random variable α with density $f_\alpha(t)$ and expected value $|\log(1-\varepsilon)|$, independent of objects constructed so far. For every $\delta>0$ there exists $c_3>0$ so small that for any interval $[t,t+c_3]$ and any $s_1,s_2\in[t,t+c_3]$, we have $f_\alpha(s_1)/f_\alpha(s_2)\in(1-\delta,1+\delta)$. We generate an integer-valued random variable N, such that $\mathbb{P}(N=j)=\mathbb{P}(\alpha\in[j2\pi/\mu_{0,k},(j+1)2\pi/\mu_{0,k}])$ for $j\ge 0$. We consider a solution to (4.61) with $\widehat{A}^{k,\varepsilon}_0=\arg X^k_{T_m}+N2\pi/\mu_{0,k}$. We generate a random variable α' with the same distribution as α conditioned to be in $[N,N+1)$. Note that we can take $\delta>0$ so small and then let k be so large that, in view of (4.55) and (4.62), the distribution of $\exp(i\widehat{A}^{k,\varepsilon}_{\alpha'-N})$ is arbitrarily close to $\nu/2$.

We generate an excursion $\bar{\mathrm{e}}^{m+1}$ with the (probability) distribution $H^0(\cdot\mid\exists t\in[0,\zeta):\mathrm{e}_t\in\mathcal{B}(0,1-\varepsilon))$. We let $\widehat{\mathrm{e}}^{m+1}_t=\exp(i\widehat{A}^{k,\varepsilon}_{\alpha'-N})\,\bar{\mathrm{e}}^{m+1}_t$.

In view of the preceding remarks, the distribution of $\widehat{\mathrm{e}}^{m+1}_0$ is arbitrarily close to $\nu/2$, conditional on the trajectories of $\mathrm{e}^1,\dots,\mathrm{e}^m$, if k is arbitrarily large. According to our construction, the joint distribution of $(\mathrm{e}^1, \dots, \mathrm{e}^m, \widehat{\mathrm{e}}^{m+1})$ is the same as that of $(\mathrm{e}^1, \dots, \mathrm{e}^m, \mathrm{e}^{m+1})$. We conclude that for any fixed $\varepsilon \in (0,1)$ and n, the joint distribution of $(\mathrm{e}^1_0, \mathrm{e}^2_0, \dots, \mathrm{e}^n_0)$ converges to that of a sequence of n i.i.d. random variables with distribution $\nu/2$, as $k \to \infty$. This completes the proof of part (i) of the theorem.

(ii) We will generalize Example 3.14. Suppose that h is positive on $\overline{D}_*$, harmonic in D_* and Lipschitz on $\overline{D}_*$. Then $1/h$ is Lipschitz on $\overline{D}_*$. Set $h_k(z) = h((1-1/k)z)$ and suppose $\mu_{0,k} \to \infty$. Then $(h_k, \mu_{0,k}) \leftrightarrow \theta_k \in \mathcal{T}$ as in Theorem 2.1, satisfy the assumptions of part (i) and the conclusions of that part of the theorem with the given h. $\square$

Proof of Theorem 3.15. (i) Suppose that X_0 has the stationary distribution with density h. Then for every $t > 0$,

$$\begin{aligned} \mathbb{E}\left[c(t)\right] &= \mathbb{E}\left[\int_0^t |f'(X_s)|^2 ds\right] &= \int_0^t \mathbb{E}\left[|f'(X_s)|^2\right] ds = \int_0^t \int_{D_*} |f'(x)|^2 h(x) dx ds \\ &= t \int_{D_*} |f'(x)|^2 h(x) dx &= t \int_D \bar{h}(x) dx < \infty. \end{aligned}$$

It follows that under the stationary distribution, $\zeta = \infty$, a.s. This implies that $\zeta = \infty$, $\mathbb{P}_x$-a.s., for almost all $x \in D_*$.

Consider an $x \in D_*$ and $r > 0$ so small that $\overline{\mathcal{B}(x,r)} \subset D_*$. The exit distributions from $\mathcal{B}(x,r)$ are mutually absolutely continuous for any two points $y, z \in \mathcal{B}(x,r)$. Let T be the exit time from $\mathcal{B}(x,r)$. It is easy to see that $c(T) < \infty$, $\mathbb{P}_y$-a.s., for every $y \in \mathcal{B}(x,r)$. Since $\zeta = \infty$, $\mathbb{P}_y$-a.s., for at least one $y \in \mathcal{B}(x,r)$, it follows that this claim holds for all $y \in \mathcal{B}(x,r)$. The claim holds for all balls such that $\overline{\mathcal{B}(x,r)} \subset D_*$ so $\zeta = \infty$, $\mathbb{P}_y$-a.s., for all $y \in D_*$.

(ii) This part follows easily from conformal invariance of Brownian motion killed upon leaving a domain.

(iii) This claim follows from the interpretation of the stationary distribution as the long time occupation measure, the definition of $\widehat{h}$ and the "clock" $c(t)$. We sketch the easy argument. For an arbitrarily small $\varepsilon > 0$ and $x, y \in D_*$ we can find $r > 0$ so small that

$$\begin{aligned} &\lim_{t\to\infty} \frac{\int_0^t \mathbf{1}_{\{Y_t \in \mathcal{B}(f(x),r)\}} ds}{\int_0^t \mathbf{1}_{\{Y_t \in \mathcal{B}(f(y),r))\}} ds} \le \lim_{t\to\infty} \frac{\sup_{z \in f^{-1}(\mathcal{B}(f(x),r))} |f'(z)|^2 \int_0^t \mathbf{1}_{\{X_t \in f^{-1}(\mathcal{B}(f(x),r))\}} ds}{\inf_{z \in f^{-1}(\mathcal{B}(f(y),r))} |f'(z)|^2 \int_0^t \mathbf{1}_{\{X_t \in f^{-1}(\mathcal{B}(f(y),r))\}} ds} \\ &\le \lim_{t\to\infty} \frac{\sup_{z \in f^{-1}(\mathcal{B}(f(x),r))} |f'(z)|^2 (1+\varepsilon) |f'(x)|^{-2} \int_0^t \mathbf{1}_{\{X_t \in \mathcal{B}(x,r)\}} ds}{\inf_{z \in f^{-1}(\mathcal{B}(f(y),r))} |f'(z)|^2 (1-\varepsilon) |f'(y)|^{-2} \int_0^t \mathbf{1}_{\{X_t \in \mathcal{B}(y,r)\}} ds} \\ &\le \lim_{t\to\infty} \frac{\sup_{z \in f^{-1}(\mathcal{B}(f(x),r))} |f'(z)|^2 (1+\varepsilon) |f'(x)|^{-2} \sup_{z \in \mathcal{B}(x,r)} h(z)}{\inf_{z \in f^{-1}(\mathcal{B}(f(y),r))} |f'(z)|^2 (1-\varepsilon) |f'(y)|^{-2} \inf_{z \in \mathcal{B}(y,r)} h(z)}. \end{aligned}$$

If we let $\varepsilon, r \to 0$ then the right hand side converges to $h(x)/h(y)$. Hence, the limsup of the left hand side is at most $h(x)/h(y)$. A similar argument shows that the liminf of the left hand side is at least $h(x)/h(y)$. This implies that the stationary density for Y is proportional to $h \circ f^{-1}$. Hence, it must be equal to $\widehat{h}$.

(iv) It follows from the definition of the "clock" $c(t)$ and the ergodic theorem that, a.s.,

$$\lim_{t\to\infty} \frac{c(t)}{t} = \int_{D_*} |f'(x)|^2 h(x) dx = \|\bar{h}\|_{L^1(D)}.$$

We have already proved (3.3). That claim and the above formula imply for $z = f(0)$,

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathbf{arg}^*(Y_t - z)}{t} &= \lim_{t\to\infty} \frac{\mathbf{arg}^* X_{c^{-1}(t)}}{t} = \lim_{t\to\infty} \frac{\mathbf{arg}^* X_t}{c(t)} = \lim_{t\to\infty} \frac{\mathbf{arg}^* X_t}{t} \cdot \frac{t}{c(t)} \\ &= \lim_{t\to\infty} \frac{\mathbf{arg}^* X_t}{t} \lim_{t\to\infty} \frac{t}{c(t)} = \frac{\mu_0}{\|\bar{h}\|_{L^1(D)}} = \frac{\mu(0)}{\|\bar{h}\|_{L^1(D)}}.\end{aligned} \tag{4.63}$$

Next we prove (3.4). Suppose that $f = \tau$ is a one-to-one analytic map of D_* onto D_* such that $\tau(0) = z$, as in Lemma 2.3. Then τ is a Möbius transformation. Let $\widehat{h} = h \circ \tau / \|h \circ \tau\|_1$, $\widehat{\mu}_0 = \mu(z)/\|h\circ\tau\|_1$, and $\widehat{\theta} = \theta\circ\tau$. Then by Lemma 2.3, $\widehat{\theta} \leftrightarrow (\widehat{h}, \widehat{\mu}_0)$. If $\bar{h} = \widehat{h}\circ\tau^{-1} = h/\|h\circ\tau\|_1$ then

$$\|\bar{h}\|_1 = 1/\|h \circ \tau\|_1.$$

By (4.63)

$$\lim_{t\to\infty} \frac{\arg^*(X_t - z)}{t} = \frac{\widehat{\mu}_0}{\|\bar{h}\|_1} = \widehat{\mu}_0 \|h \circ \tau\|_1 = \mu(z). \tag{4.64}$$

Finally, we prove (3.23) in full generality along the same lines as in (4.63). For any $z \in D$, by (3.4),

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathbf{arg}^*(Y_t - z)}{t} &= \lim_{t\to\infty} \frac{\mathbf{arg}^*(X_{c^{-1}(t)} - f^{-1}(z))}{t} = \lim_{t\to\infty} \frac{\mathbf{arg}^*(X_t - f^{-1}(z))}{c(t)} \\ &= \lim_{t\to\infty} \frac{\mathbf{arg}^*(X_t - f^{-1}(z))}{t} \cdot \frac{t}{c(t)} = \lim_{t\to\infty} \frac{\mathbf{arg}^*(X_t - f^{-1}(z))}{t} \lim_{t\to\infty} \frac{t}{c(t)} = \frac{\mu(f^{-1}(z))}{\|\bar{h}\|_1}.\end{aligned}$$

(v) Let θ correspond to (h, μ_0). Let Y be constructed as in (3.20)-(3.22). Then it is easy to see that Y satisfies conditions (a) and (b) of part (v).

(vi) This follows directly from the Itô formula and Theorem 3.1. □

We now present an example showing that a conformal mapping may not always map an ORBM in one planar domain to another ORBM, in the sense of Theorem 3.15.

EXAMPLE 4.1. Let S be a two-dimensional infinite wedge with corner at the origin 0 and angle $0 < \alpha < 2\pi$. Consider $\theta_1, \theta_2 \in (-\pi/2, \pi/2)$ and suppose that each θ_k represents the angle of reflection on one of the two sides of the wedge, measured from the inward normal toward the origin 0. In Varadhan and Williams (1995), it was shown that there exists a strong Markov process that behaves like Brownian motion in the interior of the wedge and reflects instantaneously at the boundary with the oblique angle of reflection given by θ_k. This process, called obliquely reflected Brownian motion in Varadhan and Williams (1995), is characterized as the unique solution to the corresponding submartingale problem away from the vertex.

It was shown Varadhan and Williams (1995) that the process enters 0 in a finite time and then stays there forever (i.e., it cannot be continued as a Markov process beyond that time) if and only if $\beta := (\theta_1 + \theta_2)/\alpha \geq 2$. Let D be an acute triangle obtained by truncation of the infinite wedge S. Assume that θ_1 and θ_2 are such that $\beta \geq 2$, set $\theta_3 = 0$ on the edge opposite to 0, and assume that the analogues of β at the other two vertices are strictly less than 2. Let f be a conformal mapping from the unit disk D_* onto the Jordan domain D and note that it extends to a homeomorphism from $\overline{D}_*$ onto $\overline{D}$. Let $\theta(x)$ be the pre-image of the θ-function on ∂D by f. Then θ is a piecewise constant function on ∂D_* taking values in $(-\pi/2, \pi/2)$. Thus by Theorem 3.5, the ORBM X in D_* with reflection angle θ is a continuous, conservative Markov process having stationary distribution $h(x)dx$. Consequently, $Z_t = f(X_t)$ is a continuous, conservative Markov process on $\overline{D}$. The process Z is an extension of killed Brownian motion in D modulo a time change in the sense that for every $t \geq 0$ and $\tau_t = \inf\{s \geq t : Z_s \in \partial D\}$, the process $\{Z_s, s \in [t, \tau_t)\}$ is a time change of Brownian motion killed upon exiting D. Let $\widehat{\tau}_t = \inf\{s \geq t : Z_s = 0\}$ for $t \geq 0$. Then the process $\{Z_s, s \in [t, \widehat{\tau}_t)\}$ is a time change of the obliquely reflected Brownian motion in D killed upon hitting 0. More precisely, let $x_0 = f^{-1}(0)$, $\sigma_{x_0} = \inf\{t \geq 0 : X_t = x_0\}$, $c(t) = \int_0^t |f'(X_s)|^2 ds$ and $c^{-1}(t) = \inf\{s : c(s) > t\}$. Then $Y_t = f(X_{c^{-1}(t)})$, $t \in [0, \sigma_{x_0})$, is obliquely reflected Brownian motion in D killed upon hitting 0. The result in Varadhan and Williams (1995) and Theorem 3.15 imply that $c(\sigma_{x_0}) = \int_0^{\sigma_{x_0}} |f'(X_s)|^2 ds < \infty$ but $\int_0^{\sigma_{x_0}+\varepsilon} |f'(X_s)|^2 ds = \infty$ a.s. for every $\varepsilon > 0$, and that $h \circ f^{-1} \notin L^1(D)$. □

Proof of Theorem 3.17. (i) The argument given in the proof of Theorem 3.15(i) which shows that $\zeta = \infty$, a.s., applies verbatim in the present case because we have assumed that $\|\bar{h}\|_{L^1(D)} < \infty$.

Every harmonic function h_k is bounded because θ_k is continuous and takes values in $(-\pi/2, \pi/2)$. Hence, the function $\bar{h}_k := h_k \circ f^{-1}$ is also bounded. Since D is bounded, it follows that $\|\bar{h}_k\|_{L^1(D)} < \infty$. Once again, the argument given in the proof of Theorem 3.15 (i) applies and shows that $\zeta_k = \infty$, a.s., for all k.

(ii) Recall the representation of X as the Poisson point process on the space $\mathbb{R}_+ \times \mathcal{C}_{D_*}$ (see Definition 3.9). Excursion laws are conformally invariant in the sense of the transformation in (3.20)-(3.22) by (Burdzy, 1987, Prop. 10.1) so Y can be represented as a Poisson point process on $\mathbb{R}_+ \times \mathcal{C}_D$. In other words, Y is an ERBM and it only remains to identify the corresponding $(\bar{\nu}(dx), \bar{H}^x)_{x \in \partial D}$. We can arbitrarily set the excursion intensity $\bar{\nu}$ to be $\bar{\nu}(A) = \nu(f^{-1}(A))$ for $A \subset \partial D$, in view of Remark 3.10 (ii).

We will find the matching normalization for $\bar{H}^x$. Fix some $z \in D$ and suppose that $r > 0$ is very small. The Green function $G_x(\,\cdot\,)$ in D has the property that

$$\lim_{r\to 0} \frac{\inf_{y \in \partial \mathcal{B}(z,r))} G_y(z)}{\sup_{y \in \partial \mathcal{B}(z,r))} G_y(z)} = \lim_{r\to 0} \frac{\inf_{y \in \partial \mathcal{B}(z,r))} G_y(z)}{|\log r|} = 1. \tag{4.65}$$

Let T_A denote the hitting time of A. Recall that $G_x(\,\cdot\,)$ is the density of the expected occupation time for Brownian motion in D killed upon exiting from D. Also, by Remark 3.10 (v), the density of the expected occupation time for $\bar{H}^x$ is $\bar{c}_x K_x(\,\cdot\,)$. Hence, for $x \in \partial D$, by the strong Markov property of $\bar{H}^x$,

$$\bar{c}_x K_x(z) = \int_{\partial\mathcal{B}(z,r)} G_y(z) \bar{H}^x(X(T_{\partial\mathcal{B}(z,r)}) \in dy).$$

This and (4.65) imply that, as $r \to 0$,

$$|\log r| \bar{H}^x(T_{\partial\mathcal{B}(z,r)} < \infty) = \bar{c}_x K_x(z) + o(1). \tag{4.66}$$

An analogous formula holds for excursion laws H^x in D_*, with the corresponding constants c_x equal to each other, by rotation invariance. Let $N(dx, z, r, D, t)$ be the number of excursions of the ERBM in D (here D can be also D_*), which started from $dx \subset \partial D$ before time t and hit $\partial\mathcal{B}(z,r)$ before their lifetime. It is easy to see that

$$\lim_{r\downarrow 0, \varepsilon\downarrow 0} \lim_{t\to\infty} \frac{N(dx, z, r, D, t)}{N(dx, z, r(1+\varepsilon), D, t)} = 1. \tag{4.67}$$

By the ergodic theorem,

$$\lim_{r\to 0} \lim_{t\to\infty} \frac{N(dx, 0, r, D_*, t)}{N(dy, 0, r, D_*, t)}$$

exists and is equal to $\nu(dx)/\nu(dy)$. The fact that small balls are mapped by f onto regions very close to balls, (4.67), and the definition of Y as a transform of X imply that for Y we have

$$\lim_{r\to 0} \lim_{t\to\infty} \frac{N(dx, f(0), r, D, t)}{N(dy, f(0), r, D, t)} = \frac{\nu(f^{-1}(dx))}{\nu(f^{-1}(dy))} = \frac{\bar{\nu}(dx)}{\bar{\nu}(dy)}.$$

This in turn implies that all $\bar{c}_x$ in (4.66) must be equal to each other so, in view of Remark 3.10 (iii), we may take all of them to be equal to 1.

(iii) The processes X^k converge to X in the sense of finite dimensional distributions according to Theorem 3.12. A stronger assertion follows from the proof of that theorem. Fix some $\varepsilon > 0$ and let $\mathrm{e}^{k,n}$ be the n-th excursion of the process X^k which hits the ball $\mathcal{B}(0, 1-\varepsilon)$, and let $T_\varepsilon^{k,n}$ be the hitting time of the ball. Then the joint distributions of $\{\mathrm{e}_t^{k,n}, t \in [T_\varepsilon^{k,n}, \zeta)\}$, $n \ge 1$, $\varepsilon > 0$, $\varepsilon \in \mathbb{Q}$, converge as $k \to \infty$, in the Skorokhod topology. By the Skorokhod lemma, we can assume that $\{\mathrm{e}_t^{k,n}, t \in [T_\varepsilon^{k,n}, \zeta)\}$, $n \ge 1$, $\varepsilon > 0$, $\varepsilon \in \mathbb{Q}$, converge a.s., as $k \to \infty$, in the Skorokhod topology. Hence, $X_t^k \to X_t$ for almost all $t \ge 0$ simultaneously, a.s.

The function f is Lipschitz continuous inside every disc $\mathcal{B}(0, 1-\rho)$, $\rho \in (0,1)$. This implies that for every $\varepsilon > 0$ and n, the images of the excursions $f(\mathrm{e}_t^{k,n})$ converge as $k \to \infty$, a.s., in the Skorokhod topology over their lifetimes to the corresponding excursion of Y. It will suffice to show that for every fixed $t > 0$, the clocks $c_k(t)$ converge to $c(t)$ in probability (note that the clocks are monotone functions).

Let

$$c(t) = \int_0^t |f'(X_s)|^2 ds, \qquad \text{for } t \ge 0,$$

$$Y(t) = f(X_{c^{-1}(t)}), \qquad \text{for } t \in [0, \infty), \tag{4.68}$$

$$c_k(t) = \int_0^t |f'(X_s^k)|^2 ds, \qquad \text{for } t \ge 0,$$

$$Y^k(t) = f\left(X^k_{c_k^{-1}(t)}\right), \qquad \text{for } t \in [0, \infty). \tag{4.69}$$

Then Y and Y^k's have distributions as specified in the statement of the theorem.

We will assume for a moment that X_0^k's and X_0 have stationary distributions. Let $D_\varepsilon = D_* \setminus \mathcal{B}(0, 1-\varepsilon)$. By assumption (i)

$$\int_{D_*} |f'(x)|^2 h(x) dx = \int_D h \circ f^{-1} dx < \infty. \tag{4.70}$$

By assumption D is bounded, so that $\int_{D_*} |f'|^2 dx = \text{Area}(D) < \infty$ and by the proof of Theorem 3.12, h_k converges uniformly to h. Thus

$$\sup_k \int_{D_*} |f'(x)|^2 h_k(x) dx < \infty, \tag{4.71}$$

and, moreover,

$$\lim_{\varepsilon \downarrow 0} \int_{D_\varepsilon} |f'(x)|^2 h(x) dx = 0, \tag{4.72}$$

and

$$\lim_{\varepsilon \downarrow 0} \sup_k \int_{D_\varepsilon} |f'(x)|^2 h_k(x) dx = 0. \tag{4.73}$$

For $\varepsilon > 0$ (suppressed in the notation), let

$$\bar{c}(t) = \int_0^t |f'(X_s)|^2 \mathbf{1}_{\{X_s \in D_\varepsilon\}} ds, \qquad \widehat{c}(t) = \int_0^t |f'(X_s)|^2 \mathbf{1}_{\{X_s \in \mathcal{B}(0,1-\varepsilon)\}} ds,$$
$$\bar{c}_k(t) = \int_0^t |f'(X_s^k)|^2 \mathbf{1}_{\{X_s^k \in D_\varepsilon\}} ds, \qquad \widehat{c}_k(t) = \int_0^t |f'(X_s^k)|^2 \mathbf{1}_{\{X_s^k \in \mathcal{B}(0,1-\varepsilon)\}} ds.$$

Fix some $t \geq 0$ and arbitrarily small $p_1, \delta > 0$. It follows from (4.70)-(4.73) that there exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$ and all k,

$$\begin{aligned} \mathbb{E}\left[\bar{c}(t)\right] &= \mathbb{E}\left[\int_0^t |f'(X_s)|^2 \mathbf{1}_{\{X_s \in D_\varepsilon\}} ds\right] = \int_0^t \mathbb{E}\left[|f'(X_s)|^2 \mathbf{1}_{\{X_s \in D_\varepsilon\}}\right] ds \\ &= \int_0^t \int_{D_\varepsilon} |f'(x)|^2 h(x) dx ds = t \int_{D_\varepsilon} |f'(x)|^2 h(x) dx ds < p_1 \delta, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\, \bar{c}_k(t) &= \mathbb{E} \int_0^t |f'(X_s^k)|^2 \mathbf{1}_{\{X_s^k \in D_\varepsilon\}} ds = \int_0^t \mathbb{E}\left(|f'(X_s^k)|^2 \mathbf{1}_{\{X_s^k \in D_\varepsilon\}}\right) ds \\ &= \int_0^t \int_{D_\varepsilon} |f'(x)|^2 h_k(x) dx ds = t \int_{D_\varepsilon} |f'(x)|^2 h_k(x) dx ds < p_1 \delta. \end{aligned}$$

It follows that for $\varepsilon \in (0, \varepsilon_1)$ and all k,

$$\mathbb{P}(\bar{c}(t) \geq \delta) \leq p_1 \qquad \text{and} \qquad \mathbb{P}(\bar{c}_k(t) \geq \delta) \leq p_1. \tag{4.74}$$

For almost all $s > 0$, $X_s^k \to X_s$, a.s., and $\mathbb{P}(X_s \in \partial\mathcal{B}(0, 1-\varepsilon)) = 0$. Hence, for almost all $s > 0$, a.s.,

$$\lim_{k\to\infty} |f'(X_s^k)|^2 \mathbf{1}_{\{X_s^k \in \mathcal{B}(0,1-\varepsilon)\}} = |f'(X_s)|^2 \mathbf{1}_{\{X_s \in \mathcal{B}(0,1-\varepsilon)\}},$$

and, therefore, by the bounded convergence theorem, a.s.,

$$\lim_{k\to\infty} \widehat{c}_k(t) = \lim_{k\to\infty} \int_0^t |f'(X_s^k)|^2 \mathbf{1}_{\{X_s^k \in \mathcal{B}(0,1-\varepsilon)\}} ds = \int_0^t |f'(X_s)|^2 \mathbf{1}_{\{X_s \in \mathcal{B}(0,1-\varepsilon)\}} ds = \widehat{c}(t).$$

This and (4.74) imply that for every fixed $t > 0$, a.s.,

$$\lim_{k\to\infty} c_k(t) = c(t),$$

because δ and p_1 can be chosen arbitrarily close to 0.

We can remove the assumption that the processes are in the stationary distribution as in the proof of Theorem 3.15 (i).

(iv) This can be proved just as part (iii) of Theorem 3.15.

(v) Let $h^* = \widehat{h} \circ f$. Then h^* is a positive harmonic function in D_* and so $\|h^*\|_1 = \pi h^*(0) < \infty$. Let $h = h^*/\|h^*\|_1$. By assumption, h is Lipschitz continuous on $\overline{D}_*$ and strictly positive on ∂D_*. Let $h_k(z) = (1-2^{-k})^{1/2} h((1-2^{-k})z)$. Then h_k is a sequence of positive harmonic functions in D_* with L^1 norm equal to 1 and C^2 on $\overline{D}_*$, such that $h_k \to h$ uniformly on compact subsets of D_*, and both h_k and $1/h_k$ are λ-Lipschitz on ∂D_* for some $\lambda > 0$ when k is sufficiently large. Let $\mu_{0,k} = k$, and let θ_k correspond to $(h_k, \mu_{0,k})$. Let Y^k's and Y be constructed as in the statement of Theorem 3.17. Then it is easy to see that the stationary distribution for ERBM Y has density $\widehat{h}$. □

Proof of Theorem 3.18. Let $D_*^k = f^{-1}(D_k)$. It is easy to see that D_*^k converge to D_* in the sense that for every $r < 1$ there exists k_0 such that $\mathcal{B}(0, r) \subset D_*^k$ for $k \geq k_0$. Set $x_0 = f^{-1}(y_0) = f_k^{-1}(y_0)$, $a_0 = f(0)$ and $a_k = f_k(0)$. Then $a_k \to a_0$. Let $h_k = \bar{h} \circ f_k / \|\bar{h} \circ f_k\|_1 = \bar{h} \circ f_k / (\pi \bar{h}(a_k))$, and let $\theta_k \leftrightarrow (h_k, \mu_0)$. Note that h_k are smooth and bounded on $\overline{D}_*$ and therefore θ_k are smooth on ∂D_* and take values in $(-\pi/2, \pi/2)$. Let $h = \bar{h} \circ f / \|\bar{h} \circ f\|_1 = \bar{h} \circ f / (\pi \bar{h}(a_0))$, and let $\theta \leftrightarrow (h, \mu_0)$. Then h_k converges to h uniformly on compact subsets of D_* and by (2.18), $\theta_k(z)$ converges to $\theta(z)$ uniformly on compact subsets of D_*. Since the closed unit ball in $L^\infty(\partial D_*; |dx|) = L^1(\partial D_*; |dx|)^*$ is compact in the weak-* topology, it follows that θ_k converges to θ in the in the weak-* topology in $L^\infty(\partial D_*; |dx|)$. Let X^k be the solution to (2.1) corresponding to θ_k and starting from $x_0 = f^{-1}(y_0)$ and let X be constructed as in Theorem 3.5, relative to θ and also starting from $x_0 = f^{-1}(y_0)$. Let

$$c(t) = \int_0^t |f'(X_s)|^2 ds \quad \text{and} \quad Y(t) = f(X_{c^{-1}(t)}) \quad \text{for } t \in [0, \infty), \tag{4.75}$$

$$c_k(t) = \int_0^t |f_k'(X_s^k)|^2 ds \quad \text{and} \quad Y^k(t) = f_k\left(X^k_{c_k^{-1}(t)}\right) \quad \text{for } t \in [0, \infty). \tag{4.76}$$

Then Y and Y^k's have distributions as specified in the statement of the theorem.

We will assume for a moment that X_0^k's and X_0 have stationary distributions. According to Theorem 3.5 (i), the processes $\{X_s^k, 0 \le s \le t\}$ converge weakly to $\{X_s, 0 \le s \le t\}$ in $M_1^{\mathcal{T}}$ topology. By the Skorokhod theorem, we can assume that all these processes are defined on the same probability space and $\{X_s^k, 0 \le s \le t\}$ converge almost surely to $\{X_s, 0 \le s \le t\}$ in $M_1^{\mathcal{T}}$ topology.

Let $D_\varepsilon = D_* \setminus \mathcal{B}(0, 1-\varepsilon)$. We have

$$\int_{D_*} |f'(x)|^2 h(x) dx = \frac{1}{\pi \bar{h}(a_0)} \int_D \bar{h} dx < \infty, \tag{4.77}$$

$$\sup_k \int_{D_*} |f_k'(x)|^2 h_k(x) dx = \frac{1}{\pi \bar{h}(a_k)} \sup_k \int_{D_k} \bar{h} dx < \infty, \tag{4.78}$$

and, moreover, as in (4.72) and (4.73)

$$\lim_{\varepsilon \downarrow 0} \int_{D_\varepsilon} |f'(x)|^2 h(x) dx = 0, \tag{4.79}$$

$$\lim_{\varepsilon \downarrow 0} \sup_k \int_{D_\varepsilon} |f_k'(x)|^2 h_k(x) dx = 0. \tag{4.80}$$

For $\varepsilon > 0$ (suppressed in the notation), let

$$\bar{c}(t) = \int_0^t |f'(X_s)|^2 \mathbf{1}_{\{X_s \in D_\varepsilon\}} ds, \qquad \widehat{c}(t) = \int_0^t |f'(X_s)|^2 \mathbf{1}_{\{X_s \in \mathcal{B}(0,1-\varepsilon)\}} ds,$$
$$\bar{c}_k(t) = \int_0^t |f_k'(X_s^k)|^2 \mathbf{1}_{\{X_s^k \in D_\varepsilon\}} ds, \qquad \widehat{c}_k(t) = \int_0^t |f_k'(X_s^k)|^2 \mathbf{1}_{\{X_s^k \in \mathcal{B}(0,1-\varepsilon)\}} ds.$$

Fix some $t \ge 0$ and arbitrarily small $p_1, \delta > 0$. It follows from (4.79)-(4.80) that there exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$ and all k,

$$\begin{aligned}\mathbb{E}[\bar{c}(t)] &= \mathbb{E}\left[\int_0^t |f'(X_s)|^2 \mathbf{1}_{\{X_s \in D_\varepsilon\}} ds\right] = \int_0^t \mathbb{E}\left[|f'(X_s)|^2 \mathbf{1}_{\{X_s \in D_\varepsilon\}}\right] ds \\ &= \int_0^t \int_{D_\varepsilon} |f'(x)|^2 h(x) dx ds = t \int_{D_\varepsilon} |f'(x)|^2 h(x) dx ds < p_1 \delta,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[\bar{c}_k(t)] &= \mathbb{E}\left[\int_0^t |f_k'(X_s^k)|^2 \mathbf{1}_{\{X_s^k \in D_\varepsilon\}} ds\right] = \int_0^t \mathbb{E}\left[|f_k'(X_s^k)|^2 \mathbf{1}_{\{X_s^k \in D_\varepsilon\}}\right] ds \\ &= \int_0^t \int_{D_\varepsilon} |f_k'(x)|^2 h_k(x) dx ds = t \int_{D_\varepsilon} |f_k'(x)|^2 h_k(x) dx ds < p_1 \delta.\end{aligned}$$

It follows that for $\varepsilon \in (0, \varepsilon_1)$ and all k,

$$\mathbb{P}(\bar{c}(t) \ge \delta) \le p_1 \qquad \text{and} \qquad \mathbb{P}(\bar{c}_k(t) \ge \delta) \le p_1. \tag{4.81}$$

For any fixed $\varepsilon > 0$, there is $k_0 \ge 1$ such that

$$\sup_{x \in \mathcal{B}(0,1-\varepsilon)} \left(|f'(x)|^2 h(x) \vee \sup_{k \ge k_0} |f_k'(x)|^2 h_k(x) \right) < \infty. \tag{4.82}$$

For every fixed $s>0$, $X^k_s \to X_s$, a.s., and $\mathbb{P}(X_s \in \partial \mathcal{B}(0,1-\varepsilon))=0$. Hence, for every fixed $s>0$, a.s.,

$$\lim_{k\to\infty} |f_k'(X^k_s)|^2 \mathbf{1}_{\{X^k_s \in \mathcal{B}(0,1-\varepsilon)\}} = |f'(X_s)|^2 \mathbf{1}_{\{X_s \in \mathcal{B}(0,1-\varepsilon)\}},$$

and, therefore, by the bounded convergence theorem, a.s.,

$$\lim_{k\to\infty} \widehat{c}_k(t) = \lim_{k\to\infty} \int_0^t |f_k'(X^k_s)|^2 \mathbf{1}_{\{X^k_s \in \mathcal{B}(0,1-\varepsilon)\}} ds = \int_0^t |f'(X_s)|^2 \mathbf{1}_{\{X_s \in \mathcal{B}(0,1-\varepsilon)\}} ds = \widehat{c}(t).$$

This and (4.81) imply that for every fixed $t>0$, a.s.,

$$\lim_{k\to\infty} c_k(t) = c(t), \tag{4.83}$$

because δ and p_1 can be chosen arbitrarily close to 0.

It follows easily from the definition (3.9) of convergence in $M_1^{\mathfrak{T}}$ topology and continuity of f on $\overline{D}_*$ that convergence of X^k to X in $M_1^{\mathfrak{T}}$ topology implies convergence of $f(X^k)$ to $f(X)$ in $M_1^{\mathfrak{T}}$ topology. This is because the transformation f affects only the first components of the pairs $(y_n(s), t_n(s))$ and $(y(s), t(s))$ in (3.9). When the clocks are changed, the second components are affected as well. Then we use (4.83) to conclude that Y^k converge to Y in $M_1^{\mathfrak{T}}$ topology.

We can remove the assumption that the processes are in the stationary distribution as in the proof of Theorem 3.15 (i). □

Proof of Theorem 3.8. Take a sequence of C^2 functions $\theta_k : \partial D_* \to (-\pi/2, \pi/2)$ that converges to $\theta \in \mathfrak{T}$ in weak-* topology as elements of the dual space of $L^1(\partial D_*)$. Let X^k be ORBM on D_* that satisfies (3.10). By Theorem 3.5(i), X^k converges weakly in $M_1^{\mathfrak{T}}$-topology to X, so does $f(X^k)$ to $f(X)$. Define

$$c_k(t) = \int_0^t |f'(X^k_s)|^2 ds \quad \text{and} \quad c(t) = \int_0^t |f'(X_s)|^2 ds.$$

By an argument similar to that proving (4.83), we can show that $\lim_{k\to\infty} c_k(t) = c(t)$ a.s. for every fixed $t>0$. Consequently by the argument as in the second to the last paragraph in the proof of Theorem 3.18, $f\left(X^k_{c_k^{-1}(t)}\right)$ converges weakly in $M_1^{\mathfrak{T}}$-topology to $f(X_{c^{-1}(t)})$. It is easy to see that $f(X_{c^{-1}(t)})$ has stationary distribution with density $\bar{h}$. Since f is smooth on $\overline{D}_k$ and $\theta_k \circ f^{-1}$ converges to $\theta \circ f^{-1} \in \mathfrak{T}$ in weak-* topology as elements of the dual space of $L^1(\partial D_*)$, it follows from Theorem 3.5 that $f(X_{c^{-1}(t)})$ is the ORBM on D_* with reflection angle $\theta \circ f^{-1}$. □

Proof of Theorem 3.19. This theorem can be proved just like Theorem 3.18. All we have to check is whether the following claims hold: (4.77), (4.78), (4.79), (4.80), and (4.82). They are all easily seen to hold in the present context. □

EXAMPLE 4.2. We will sketch an example of a bounded domain D, an oblique angle of reflection θ and the corresponding ORBM with a stationary measure whose density h is not in $L^1(D)$. The construction is a typical fractal-type argument; a construction similar in spirit can be found in Section 4 of Bass and Burdzy (1992). We will not supply a formal proof because it would require a lot of space and the claim is rather specialized.

Let $D_0 = (0,1)^2$, and for $k \geq 1$ and small $r_k \in (0, 2^{-k-2})$ (to be specified later), let

$$\begin{aligned} D_k &= \mathcal{B}(2^{-k} - i2^{-k}, 2^{-k-2}), \\ D'_k &= (2^{-k} - r_k, 2^{-k} + r_k) \times (-2^{-k}, 2^{-k}), \\ D &= D_0 \cup \bigcup_{k \geq 1} (D_k \cup D'_k). \end{aligned}$$

The boundary ∂D is smooth except for a countable number of points. We will specify the reflection angle relative to the inward normal vector $\mathbf{n}$ at each boundary point where $\mathbf{n}$ is well defined. For all points $x \in \partial D \cap (\partial D_0 \cup \partial D_k)$, $k \geq 0$, we let $\theta(x) = 0$. In other words, the reflection is in the normal direction at the points on the boundary of the square D_0 and on the (arcs of the) circles ∂D_k.

It remains to define the angle of reflection for the part of ∂D which lies on the sides of very thin channels D'_k. To make the example simple, we let the angle of reflection be $\pi/2$ or $-\pi/2$, at $x \in \partial D \cap \partial D'_k$, $k \geq 1$, so that the reflected process is pushed down towards D_k. It would be more accurate to say that the process is teleported to D_k if it hits the side of a channel $\partial D \cap \partial D'_k$ because it has a jump that takes it to ∂D_k.

Heuristically speaking, the ratio of the average amounts of time spent by ORBM in D_k and D_0 can be made arbitrarily large by making r_k sufficiently small. The reason is that ORBM will jump to D_k when it hits the boundary of D'_k. Going the other way is much harder—the process has to go though the very thin channel connecting D_k and D_0 without hitting the sides of the channel. Let a_k be the ratio of the average amounts of time spent by ORBM in D_k and D_0. If we make all $a_k \geq 1$ then $\sum_{k \geq 1} a_k = \infty$ and it follows that there is no stationary probability distribution for ORBM. Every stationary measure has to have infinite mass.

It is clear that the ORBM described above is well defined as long as it does not hit $(0,0)$. An elementary argument can be used to show that the ORBM will not hit $(0,0)$ at a finite time, a.s., if we make the channels sufficiently thin (i.e., r_k's sufficiently small). □

Proof of Theorem 3.20. Parts (i) and (ii) are special cases of Theorems 1 and 2 of Aikawa (2000).

For part (iii), let D be the image of the unit disk by the map $F(z) = \sqrt{1-z}$ and let $h(w) = \operatorname{Re}((1+z)/(1-z))$ where $z = F^{-1}(w)$. Then for the region C in the disk given by $1 - |z|^2 > |1-z|$ (an approximate cone),

$$\int_D h(w)dw = \int_C \operatorname{Re}((1+z)/(1-z))|F'(z)|^2 dz \geq \int_C |1-z|^{-2} dz/4,$$

since $\operatorname{Re}((1+z)/(1-z)) = (1-|z|^2)/|1-z|^2$. This latter integral is infinite by integrating in polar coordinates centered at $z = 1$. □

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