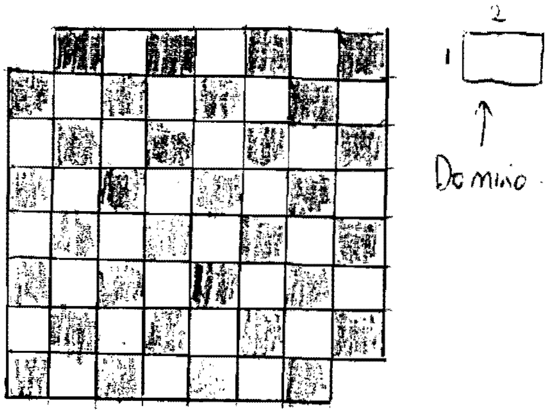
Domino Tiling and Determinants



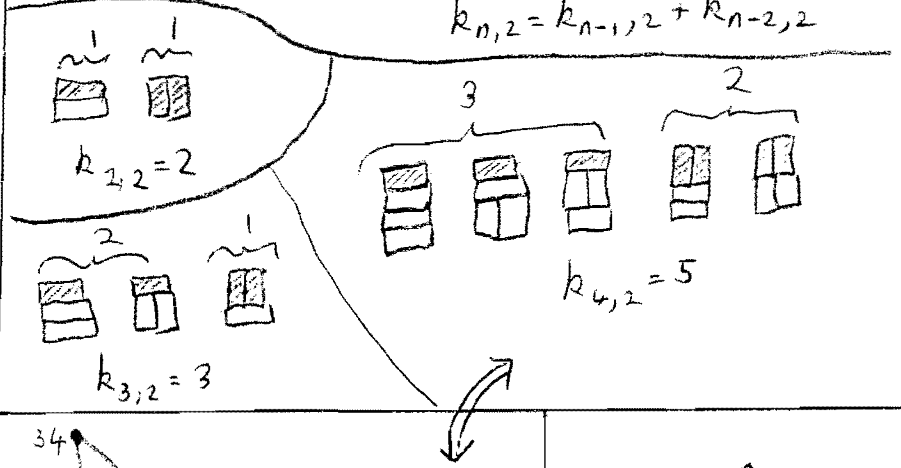
Its impossible to tile this truncated chessboard with 2×1 domino tiles. Why?

$k_{n,m}$ = # of ways of tiling $n\times m$ rectangle with 2×1 tiles

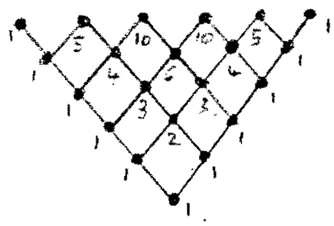
$k_{n,m}=0$ if n,m both odd. Why?

$k_{n,2}$ = Fibonacci $\quad k_{0,2}=1 \quad k_{1,2}=1$

$k_{n,2}=k_{n-1,2}+k_{n-2,2}$



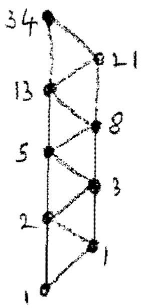
Counting Paths in Graphs



Path counting

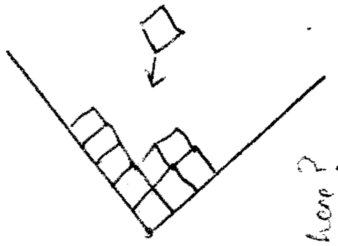
Pascal's Δ

(Binomial coefficients)

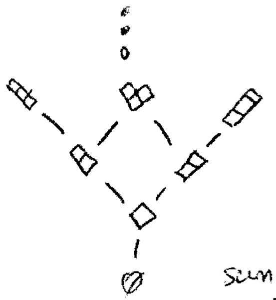


Path counting

(Fibonacci again)



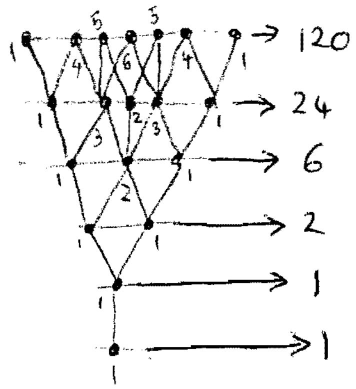
Tetris like shapes

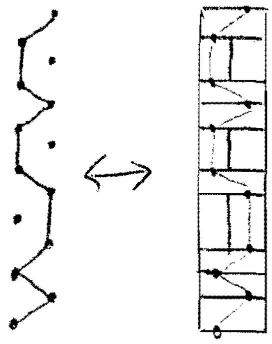


Young's lattice

What's the pattern here?

sums of squares

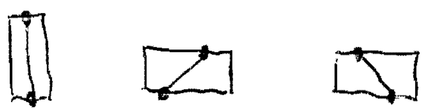




How to explain the coincidence between $k_{n,2}$ and paths in the Fibonacci graph?

1–1 correspondence between paths and tilings

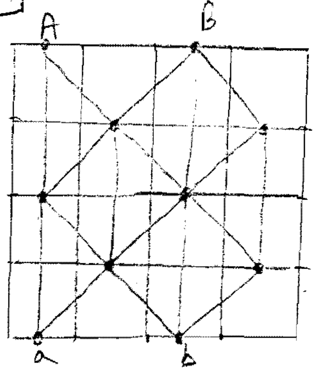
Rule:

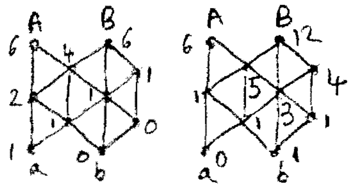


$$k_{2n,2m}=4^{mn}\prod_{j=1}^{m}\prod_{k=1}^{n}\left(\cos^2\frac{j\pi}{2m+1}+\cos^2\frac{k\pi}{2n+1}\right)$$

Kastelyn's formula (1961)

$\boxed{4\times 4}$





Tilings $\longleftrightarrow$ non-intersecting paths $\left\{ \begin{matrix} a \longleftrightarrow A \\ b \longleftrightarrow B \end{matrix} \right\}$

$$= \#\text{paths} \left\{ \begin{matrix} a \longleftrightarrow A \\ b \longleftrightarrow B \end{matrix} \right\} - \#\text{paths} \left\{ \begin{matrix} a \longleftrightarrow B \\ b \longleftrightarrow A \end{matrix} \right\}$$

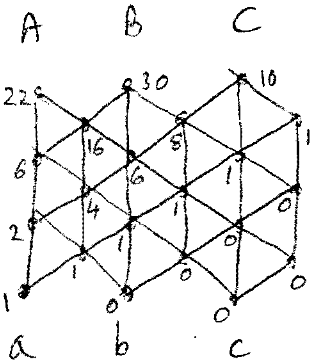
$$= \det \begin{array}{c|c} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 6 & 6 \\ 6 & 12 \end{bmatrix} \end{array} = 36$$

$6\times 12 - 6\times 6$

#Paths $a \longleftrightarrow A$ #Paths $a \longleftrightarrow B$

#Paths $b \longleftrightarrow A$ #Paths $b \longleftrightarrow B$

$\boxed{6\times 6}$



Tilings $\longleftrightarrow$ non-intersecting paths $\left\{ \begin{matrix} a \longleftrightarrow A \\ b \longleftrightarrow B \\ c \longleftrightarrow C \end{matrix} \right\}$

Proof?

$$\overset{?}{=} \det \begin{array}{c|c} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 22 & 30 & 10 \\ 30 & 62 & 40 \\ 10 & 40 & 52 \end{bmatrix} \end{array} = 6728$$

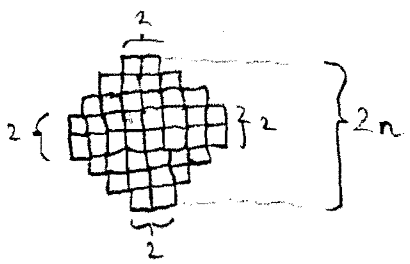
<u>Determinant</u> — try googling it!

The method works for all sizes of rectangle!

Questions

① What is k_{88}?

② How many domino tilings does the Aztec diamond of height $2n$ have?



Aztec diamond of height 8

$$\det \begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix} = ptx + quv + rsw - rtv - qsx - puw$$

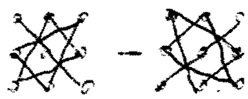


Table of $k_{m,n}$ $(m,n \le 8)$

0	1	0	1	0	1	0	1
1	2	3	5	8	13	21	34
0	3	0	11	0	41	0	153
1	5	11	36	95	281	781	2245
0	8	0	95	0	1183	0	14824
1	13	41	281	1183	6728	31529	167089
0	21	0	781	0	31529	0	1292697
1	34	153	2245	14824	167089	1292697	1??????6