1. Is there a graph on 10 vertices with degrees 1, 1, 3, 3, 6, 6, 6, 7, 8, 9?

   No. The vertex of degree 9 implies that all the other vertices must have degree ≥ 1. Also, the vertex of degree 8 implies that there can only be one vertex with degree 1, so this graph cannot exist.

2. Five points with integer coordinates are chosen on a plane. Prove that the midpoint of one of the segments joining two of these points also has integer coordinates.

   For arbitrary points in the plane with integer coordinates, there are four possibilities: both coordinates can be even, both coordinates can be odd, the first can be even and the second odd, or the first odd and the second even. Since there are four categories in the above list and five points were picked, there must be at least two points in the same category. The midpoint of the segment joining those two points has integer coordinates.

3. p.11, #15

   (a) We select 11 positive integers that are less than 29 at random. Prove that there will always be two integers selected that have a common divisor larger than 1.
   Since all picked 11 numbers are distinct, at least 10 of them are larger than 1. These ten or eleven numbers (that is, all selected numbers that are larger than 1) will be our pigeons. Our pigeonholes will be all primes that are less than 29, namely,

   $$2, 3, 5, 7, 11, 13, 17, 19, 23.$$

   Put each pigeon in the pigeonhole that corresponds to its smallest prime divisor. Since there are ≥ 10 pigeons and only 9 pigeonholes, there must be at least two selected integers that have the same prime divisor. The common divisor of these two integers is larger than 1.

   (b) Is the statement of part (a) true if we only select ten integers that are less than 29?
   NO: among the integers $1, 2, 3, 5, 7, 11, 13, 17, 19, 23$ no two have a common divisor larger than 1.

4. p.11, #20: we are given 17 points inside a regular triangle of side length one. Prove that there are two points among them whose distance is not more than 1/4.

   Split the triangle into 16 regular triangles of side 1/4 by lines that are parallel to the sides of the triangle. By Pigeon-hole Principle, one of these triangles must contain two of our 17 points. The distance between these two points is at most 1/4.

5. p. 28, #(17) Prove that for all positive integers n,

   $$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + ... + n)^2.$$

   The proof is by induction on n. If n=1, then both sides are equal to 1. Now assume that the statement is true for n, that is,

   $$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + ... + n)^2. \tag{1}$$

   and prove it for $n+1$. Indeed, since $a^2 - b^2 = (a-b)(a+b)$ and since $2(1+2+...+n) = n(n+1)$, we have

   $$\begin{aligned}((1 + 2 + ... + n) + (n + 1))^2 - (1 + 2 + ... + n)^2 &= (n + 1) \cdot (2(1 + 2 + ... + n) + (n + 1)) \\ &= (n + 1) \cdot (n(n + 1) + (n + 1)) = (n + 1)^3,\end{aligned}$$

   that is,

   $$(n + 1)^3 = ((1 + 2 + ... + n) + (n + 1))^2 - (1 + 2 + ... + n)^2 \tag{2}$$

   Adding equations (1) and (2), implies the result.

6. p. 28, #(23) Let $a_0 = 0$, $a_1 = 1$, and let $a_{n+2} = 6a_{n+1} - 9a_n$ for $n \geq 0$. Prove that $a_n = n \cdot 3^{n-1}$ for all $n \geq 0$.

   The proof is by (strong) induction on n. If $n = 0$ or $n = 1$ the statement is clearly true. Now assume that the statement is true for n and $n + 1$, and prove it for $n + 2$. Indeed, by our assumption $a_n = n \cdot 3^{n-1}$ and $a_{n+1} = (n + 1) \cdot 3^n$. Hence

   $$a_{n+2} = 6a_{n+1} - 9a_n = 6 \cdot (n + 1) \cdot 3^n - 9n \cdot 3^{n-1} = 2(n + 1) \cdot 3^{n+1} - n \cdot 3^{n+1} = (n + 2) \cdot 3^{n+1},$$

   as required.

7. p. 28, #(29) Prove that for any positive integer n, it is possible to partition any triangle T into $3n + 1$ similar triangles.

   The proof is by induction on n. If $n = 1$, then one can partition T into $3 \cdot 1 + 1 = 4$ triangles by using its three midparallel lines. We now assume that T can be partitioned into $3n + 1$ similar triangles, and prove that then it can be also partitioned into $3(n + 1) + 1 = 3n + 4$ similar triangles. Indeed take any partition of T into $3n + 1$ similar triangles, say $T_1, \ldots, T_{3n}, T_{3n+1}$. Now partition T_{3n+1} into 4 similar triangles, say $D_1, \ldots, D_4$ by using the three midparallel lines of T_{3n+1} . Each of these four triangles is similar to T_{3n+1}, and hence also to T, as T_{3n+1} is similar to T. The partition of T into $T_1, \ldots, T_{3n}, D_1, \ldots, D_4$ is then a required partition of T into $3n + 4$ similar triangles.