

1. Determine the number of graphs on 10 vertices labeled $1, 2, \dots, 10$ that contain exactly two out of the following four edges $e_1 = \{1, 2\}$, $e_2 = \{1, 3\}$, $e_3 = \{2, 3\}$, and $e_4 = \{1, 4\}$.

First, we must choose two of the four edges to be included in the graph. This leaves us with $\binom{10}{2} - 4$ possible pairs to choose the edges from, so we get $\binom{4}{2} 2^{\binom{10}{2} - 4}$ possible graphs.

2. In Never-Never-Land there is only one means of transportation: magic carpet. Twenty-one carpet lines serve the capital. A single line flies to Farville, and every other city is served by exactly 20 carpet lines. Show that it is possible to travel by magic carpet from the capital to Farville (perhaps by transferring from one carpet line to another).

Let's look at the connected component of the graph of carpet lines which includes Farville. We must prove that this component includes the capital. Suppose it does not. Then there is 1 edge starting at one vertex, and 20 edges starting at every other vertex. Therefore this connected component contains exactly one odd vertex. This is a contradiction.

3. Prove that a connected graph with $2n$ vertices of odd-degree and any number of even-degree vertices can be drawn without drawing any edge more than once and in such a way that the pencil will be lifted off the paper exactly $n - 1$ times.

The proof can be carried out using induction on n . The trick is that we prove a stronger statement, namely, that every connected **multigraph** with $2n$ vertices of odd-degree (and any number of even-degree vertices) can be drawn without drawing any edge more than once and in such a way that the pencil will be lifted off the paper exactly $n - 1$ times. The base $n = 1$ is the Euler theorem. To prove the inductive step we choose two vertices of odd degree A and B and connect them mentally with a new edge. After that the new multigraph has only $2n - 2$ odd-degree vertices, and so by inductive assumptions can be drawn in such a way that the pencil will be lifted exactly $n - 2$ times. When, in the process of drawing, we must go along the mentally added edge AB (which does not really exist), we simply lift the pencil from the paper and put it down at the other end of this edge.

4. p. 103, #(29) Let a_n be the number of compositions of n into parts that are larger than 1. Express a_n by a_{n-1} and a_{n-2} .

We claim that $a_n = a_{n-1} + a_{n-2}$. Indeed, each such composition either has its last part equal to 2 or larger than 2. In the first case, deleting the last part of a composition results in a composition of $n - 2$ all of whose parts are larger than 1. Thus there are altogether a_{n-2} such compositions. In the second case, decreasing the last part of a composition by 1 results in a composition of $n - 1$ all of whose parts are larger than 1. There are a_{n-1} such compositions. Thus the total number of allowed compositions of n is $a_{n-1} + a_{n-2}$. To completely determine the sequence a_n we need to specify initial values. These are given by $a_1 = 0$ and $a_2 = 1$.

5. p. 103, #(30) Let b_n be the number of compositions of n into parts that are larger than 2. Find a recurrence relation satisfied by b_n .

As in the previous problem, the last part of such a composition is either equal to 3 or larger than 3. In the first case, removing the last part produces an allowed composition of $n - 3$, while in the second case decreasing the last part of a composition by 1 produces an allowed composition of $n - 1$. Thus the recurrence is $b_n = b_{n-1} + b_{n-3}$ with the initial values given by $b_1 = b_2 = 0$ and $b_3 = 1$.

6. p. 199, #(28) Ten players participate at a chess tournament. Eleven games have already been played. Prove that there is a player who has played at least 3 games.

This situation can be described by a graph G : the vertices of G are players and the edges of G are between pairs of players who've already played with each other. Thus the number of vertices of G is the number of players, that is 10; the number of edges of G is the number of

games that have already been played, that is, 11; and the degree of each vertex is the number of games that the corresponding player has played. If no one has played 3 or more games, then the degree of each vertex in G would be ≤ 2 , and hence the number of edges, $|E|$ would be $\leq 1/2 \cdot |V| \cdot 2 = |V| = 10$, contradicting the fact that $|E| = 11$. Thus there is a vertex of degree at least 3, and hence there is a player who has played ≥ 3 games.

Note that the solution works equally well if G is a multigraph, so it also covers those types of chess tournaments in which a pair of players may play each other more than once. In this case we just place as many edges between two players as the number of games they've played. (But a player never plays himself, so there will be no loops in the graph.)