Algebraic combinatorics on simplicial complexes — Information sheet

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In this class we will concentrate on studying combinatorics of simplicial complexes and in particular their face numbers. A simplicial complex is a family of sets closed under inclusion. For instance, the collection of cliques of a given graph is such a complex. Described geometrically, a simplicial complex consists of vertices, edges, triangles, and higher dimensional simplices "nicely" glued together. Thus another big class of examples comes from triangulated spheres and, more generally, triangulated manifolds.

The face vector, a basic combinatorial invariant of a simplicial complex, encodes the number of faces of the complex in each dimension. A typical combinatorial problem is to characterize those integer vectors that can arise as the face vectors of a given family of simplicial complexes. (For instance, is there a graph on 100 vertices that has 1000 edges, 100000 cliques of size three, 1000000 cliques of size 4, and no larger sized cliques?) This problem, while very easy to state, turns out to be notoriously difficult, and is still wide open for many interesting classes of complexes. In most cases where solutions (or partial solutions) are known they involve a fascinating interplay of techniques from combinatorics, commutative algebra, and elementary algebraic topology.

In this course we will introduce several such techniques, among them the theory of Stanley-Reisner rings, methods of combinatorial and algebraic shifting, Alexander duality, etc.

Prerequisites: familiarity with basic commutative algebra notions (such as polynomial rings and ideals) as well as with the basics of algebraic topology (simplicial homology) will be assumed, although some of these notions will be reviewed in class.

Preliminary list of topics.

(1) Simplicial complexes and their basic combinatorial and topological invariants.
(2) Algebraic invariants of simplicial complexes via Stanley-Reisner rings.
(3) Eulerian and semi-Eulerian complexes; Dehn-Sommerville relations.
(4) Basics on polytopes; the cyclic polytope.
(5) Cohen-Macaulay rings and the Upper Bound Theorem for spheres.
(6) Reisner's and Munkres' criteria for Cohen-Macaulayness.
(7) The Upper Bound Theorem for odd-dimensional manifolds.
(8) Characterizing face numbers of simplicial complexes and Cohen-Macaulay simplicial complexes.
(9) Generic Initial ideals/ algebraic shifting.
(10) Characterizing face numbers of simplicial complexes with prescribed Betti numbers.
(11) Shellable complexes.
(12) Balanced complexes.

(13) Complexes with symmetry.
(14) Flag complexes.
(15) Characterizing face numbers of simplicial polytopes.

Books: There will be no official textbook. The material we will cover is a (very small) subset of the union of the following books (plus some research papers). All these books are on reserve in Math library.

(1) Richard P. Stanley, "Combinatorics and Commutative Algebra", second edition.
(2) Takayuki Hibi, "Algebraic Combinatorics on Convex Polytopes".
(3) Winfried Bruns & Jürgen Herzog, "Cohen-Macaulay Rings", revised first edition.
(4) Ezra Miller and Bernd Sturmfels, "Combinatorial Commutative Algebra".

In addition, the following two books

- Davis Eisenbud, "Commutative Algebra: with a View Toward Algebraic Geometry"
- James R. Munkres, "Elements of Algebraic Topology"

are excellent texts on Commutative Algebra and Simplicial Homology and you may want to consult them from time to time.

Grades: Grading will be mostly based on homework problem sets. A part of your grade may also come from presenting a research article (let's leave the specifics on such presentations for later).