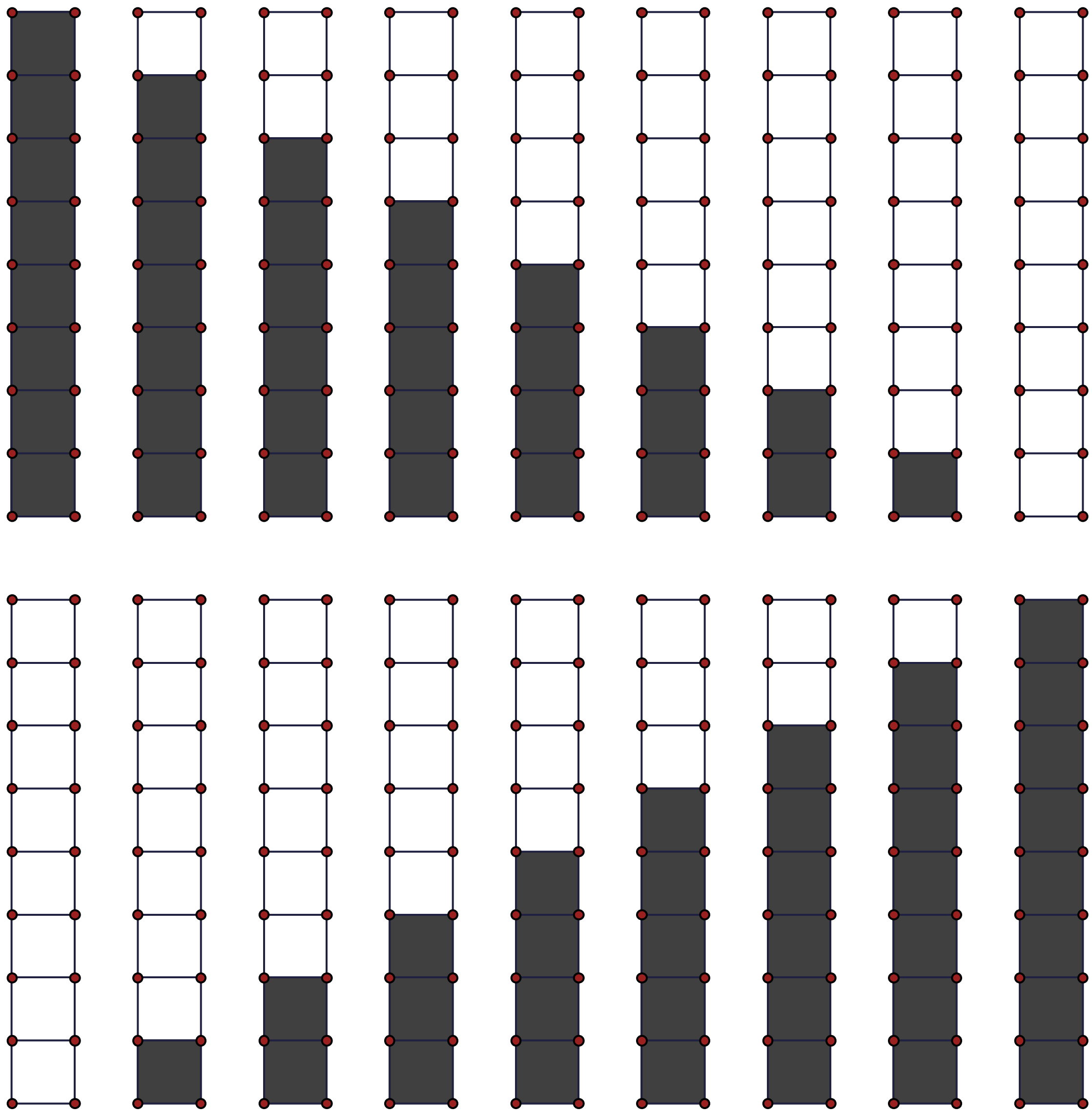
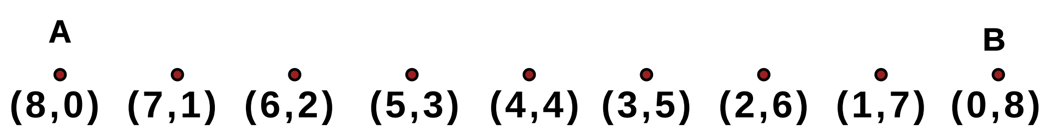
How to Represent Fluid Level in Two Containers

Suppose the total amount of fluid is 8 pints, divided between Container A and Container B. Sticking with integer amounts, here are the possible fluid levels. The top row represents A and the bottom is B.

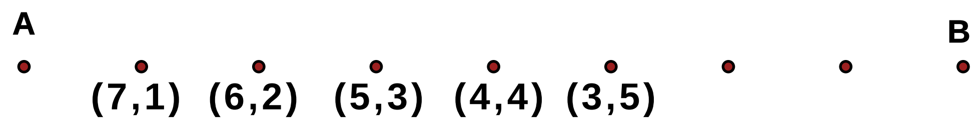


We can represent the same fluid levels by points on segment AB, dividing AB into 8 equal segments. Notice that (8,0) represents all the fluid in A and none in B, etc.



This set of 9 points represents all the possible states if each of A and B have capacity 8 or more. What if B will hold only 5 pints and A will hold only 7 pints.
Then none of the states with B volume = 6, 7, 8 are possible, so (2,6), (1,7) and (0,8) are not possible because of the volume of B. Likewise (8,0) is out because A will only hold 7 pints.

Now the set of possible volumes looks like the labeled points below.



Two-Jug Exercise:

Suppose two jugs A and B hold a total of 4 pints (with a whole number of pints in each). Using the same idea as above, represent the possible distributions of liquid in the two jugs using the same idea of a row of labeled points as above.

- How many points are there?

- Write them down and label them.

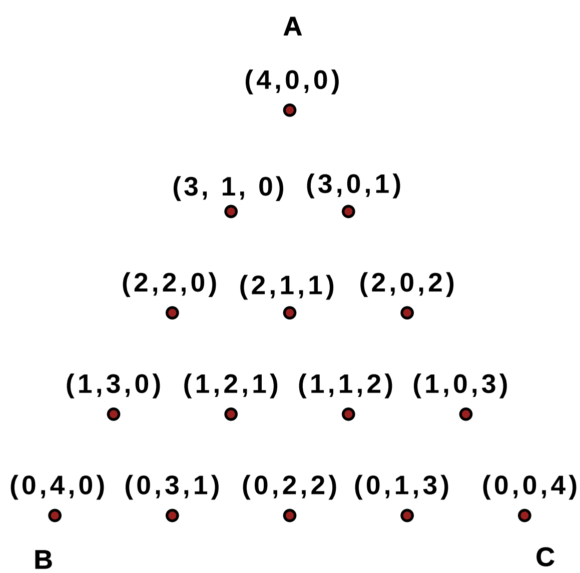
Three Jugs

Now let's consider how to represent 3 jugs. To make it simple to start with, assume that the total fluid is 4 pints.

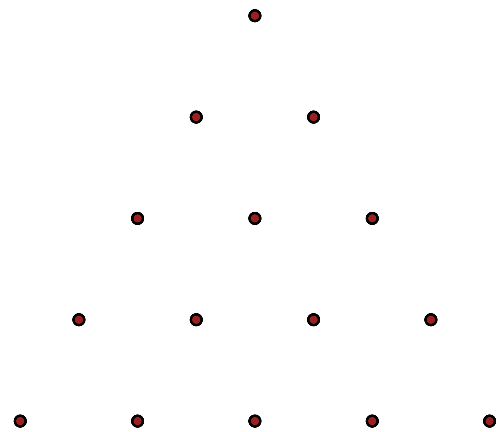
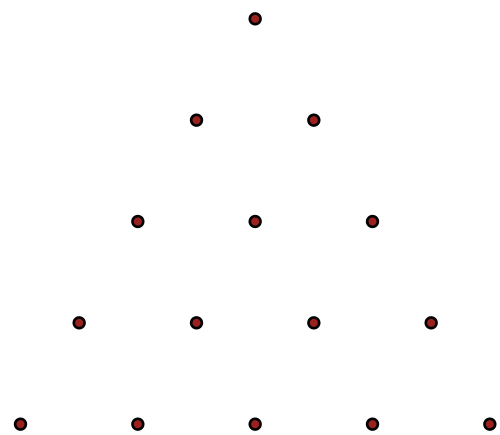
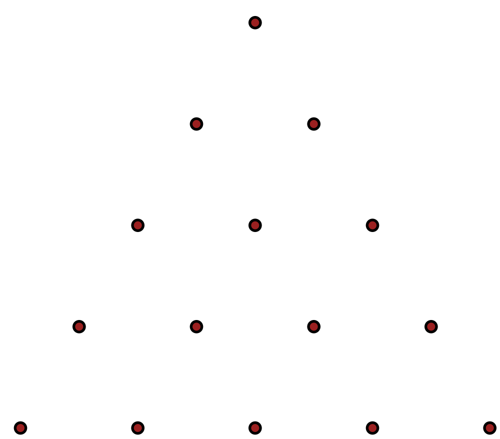
- **Case 0.** 4 pints are in A and 0 are in B or C. This would be represented by one point labeled (4, 0, 0).
- **Case 1**. 3 pints are in A and 1 pint is in B or C. There are two sub-cases here. Either B has 1 pint and C has 0 or B has 0 and C has one. We denote these cases by two points labeled (3, 1, 0) and (3, 0, 1).
- **Case 2.** 2 pints are in A and 2 pints are distributed in B or C. There are 3 sub-cases here. Either B has 2 pints and C has 0 or B has 1 and C has 1 or B has 0 and C has 2. We denote these cases by 3 points labeled (2, 2, 0), (2, 1, 1), (2, 0, 2).
- **Case 3.** 1 pint is in A and 3 pints are distributed in B or C. There are 4 sub-cases here. We denote these cases by 4 points labeled (1, 3, 0), (1, 2, 1), (1, 1, 2), (1, 0, 3).
- **Case 4.** 0 pints are in A and 4 pints are distributed in B or C. There are 5 sub-cases here. We denote these cases by 5 points labeled (0, 4, 0), (0, 3, 1), (0, 2, 2), (0, 1, 3), (0, 0, 4).

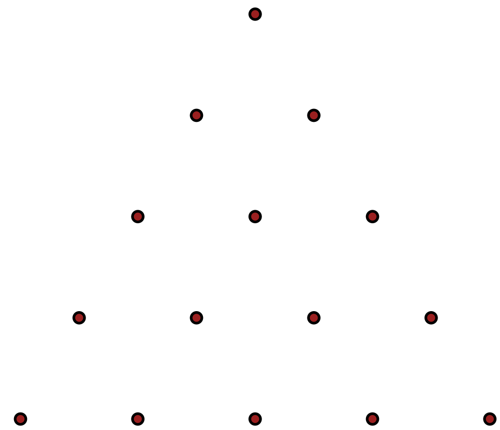
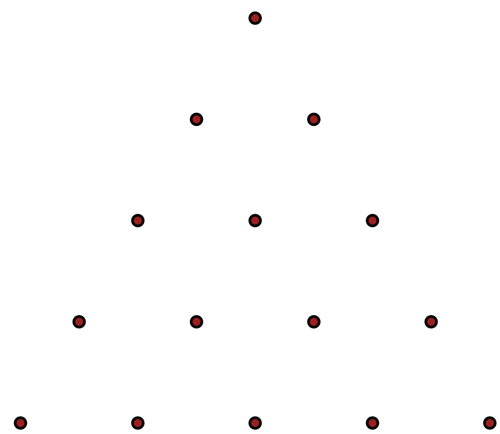
We notice two things.

1. Each case, if we ignore A, is really just a two-jug problem where the total fluid changes in each case. For example, Case 4 is just like the Two-Jug Exercise.
2. If we write down all the rows from the cases, we get a triangle like this. Each point represents one of the possible volume distributions or **STATES**.



Exploring the 3 Jug Diagram -- Learning the Moves

Here are some unlabeled copies of the triangle on the previous page. · Draw circles and label the points representing cases where all the liquid is in one jug.	
Draw separate lines connecting the points showing states where: · There are 4 pints in C · There are 3 pints in C · There are 2 pints in C · There is 1 pints in C · There are 0 pints in C	
Label the point (3, 1, 0). This is a state with 3 pints in A. · Draw an arrow from this point to the point that represents the result of emptying A into B. · Draw an arrow from this point to the point that represents the result of emptying A into C. Notice that these new points are on the boundary of the triangle. Why? If an arrow is parallel to a side, explain why this means the liquid in one of the containers is unchanged.	

Label point (1,2,1). Indicate by arrows all the possible results of pouring all or part of the liquid from one of the jugs into another one. Why are these arrows all parallel to a side? Draw little squares around the states that cannot be reached by one pour from (1,2,1). Indicate by circling the arrowheads the pours where *ALL* the liquid was poured out of one of the jugs.	
Now suppose Jug C only holds 3 pints and Jug B only holds 2 pints, but Jug A will hold 4 pints or more. · Cross out all the states that are impossible with these jugs. · Outline the set of states that are possible. · What kind of polygon is this outline?	

Three-Jug Problems:

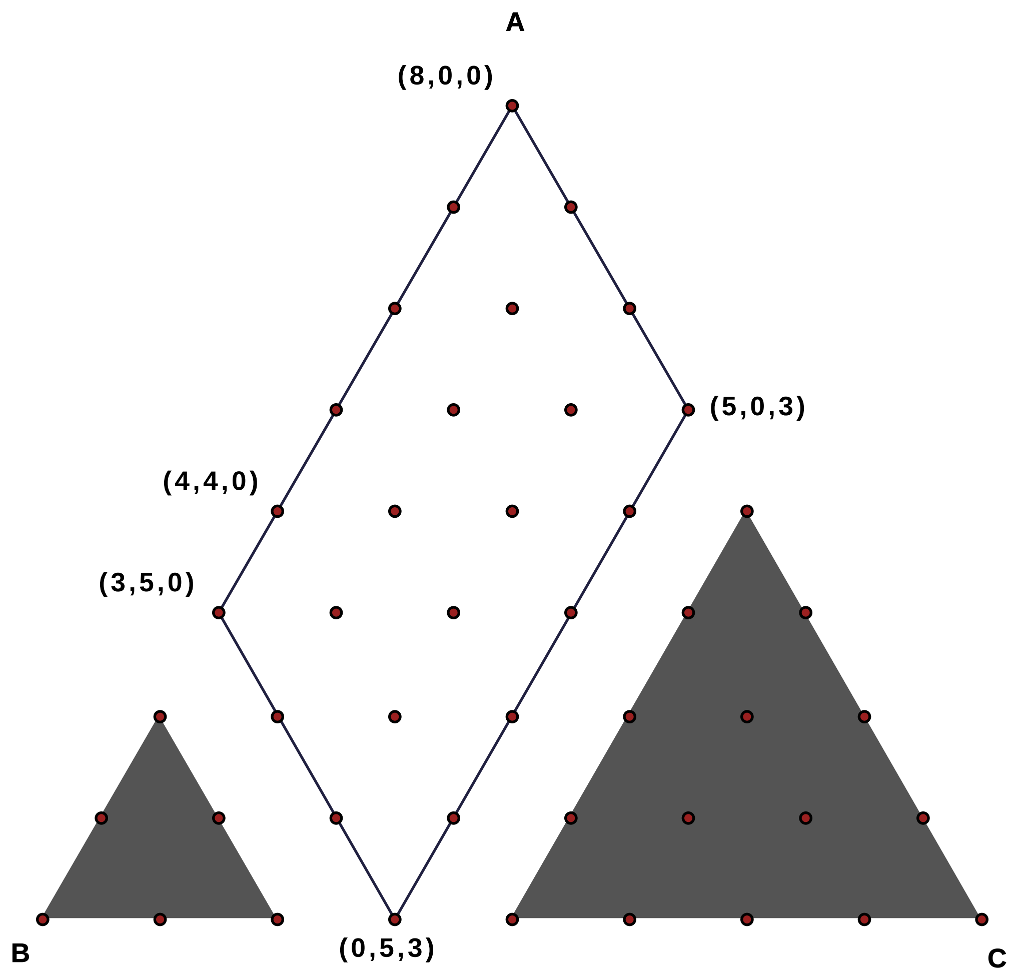
Problem

Suppose that Jug A contains 8 pints of liquid. Suppose Jug B has a capacity of 5 pints and Jug C has a capacity of 3 pints. By pouring back and forth, find a way to get 4 pints in one jug.

Note: There are no measuring lines on the jugs, so we can either empty one jug into another or pour from one until the other is full.

Solution Method

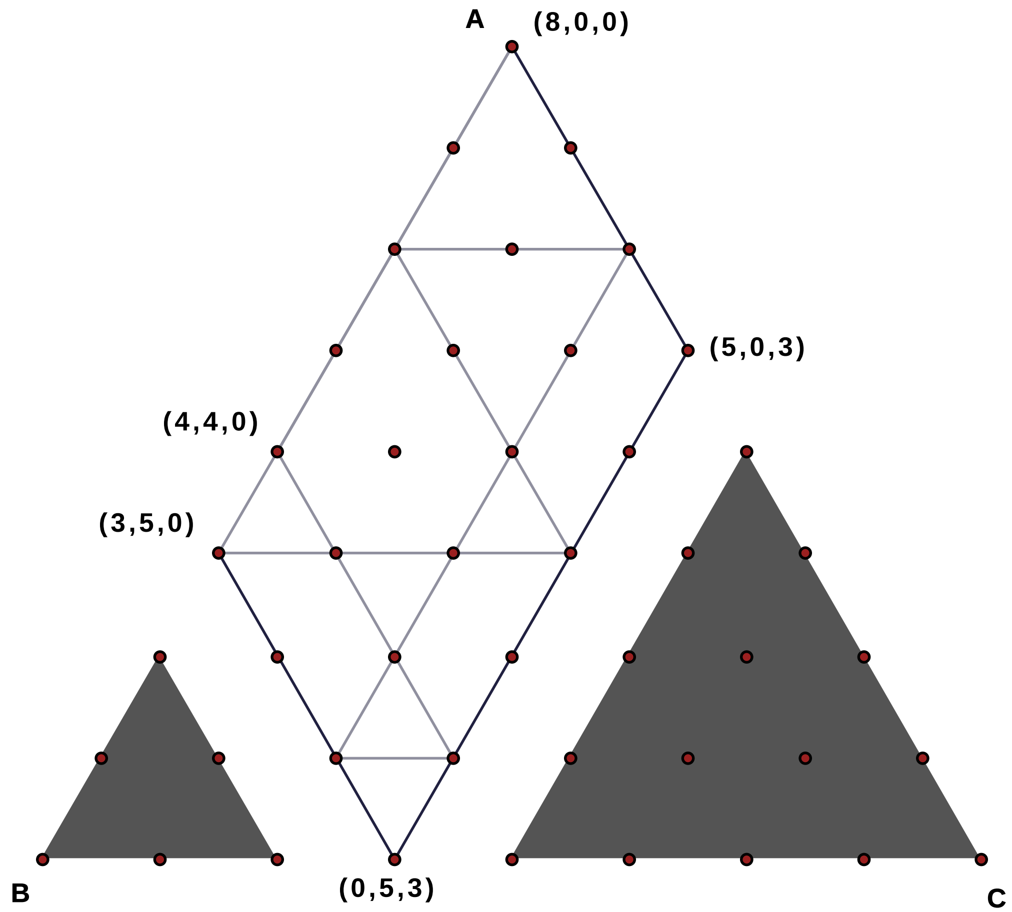
We can represent all possible states of distribution of the liquid by this triangle with the points in the shaded areas removed because of the capacity of B and C. The remaining points are bounded by a parallelogram.



We are starting at point A = (8,0,0) and want to arrive at a point whose label has a 4 in it.

- Explain why any of the legal pouring operations at any stage will end up on the boundary of the parallelogram.
- Our goal is a point whose label has a 4 in it. Which points are these? Start by circling all such points that are also on the boundary of the parallelogram. It is possible to end up at (4,4,0) if we can get to a 4.
- Indicate the legal moves from (8,0,0) and then continue tracing legal moves to see whether the goal is reached. Try to find a winning sequence of moves.
- Draw the winning sequence on the triangle on the sheet (or a scratch triangle on one of the extra sheets of dot paper).
- Write down the winning sequence so that we can more easily compare solutions.
- When you have finished, you can peek on the back to see a winning solution. What was the first move in this case? What was the second move?

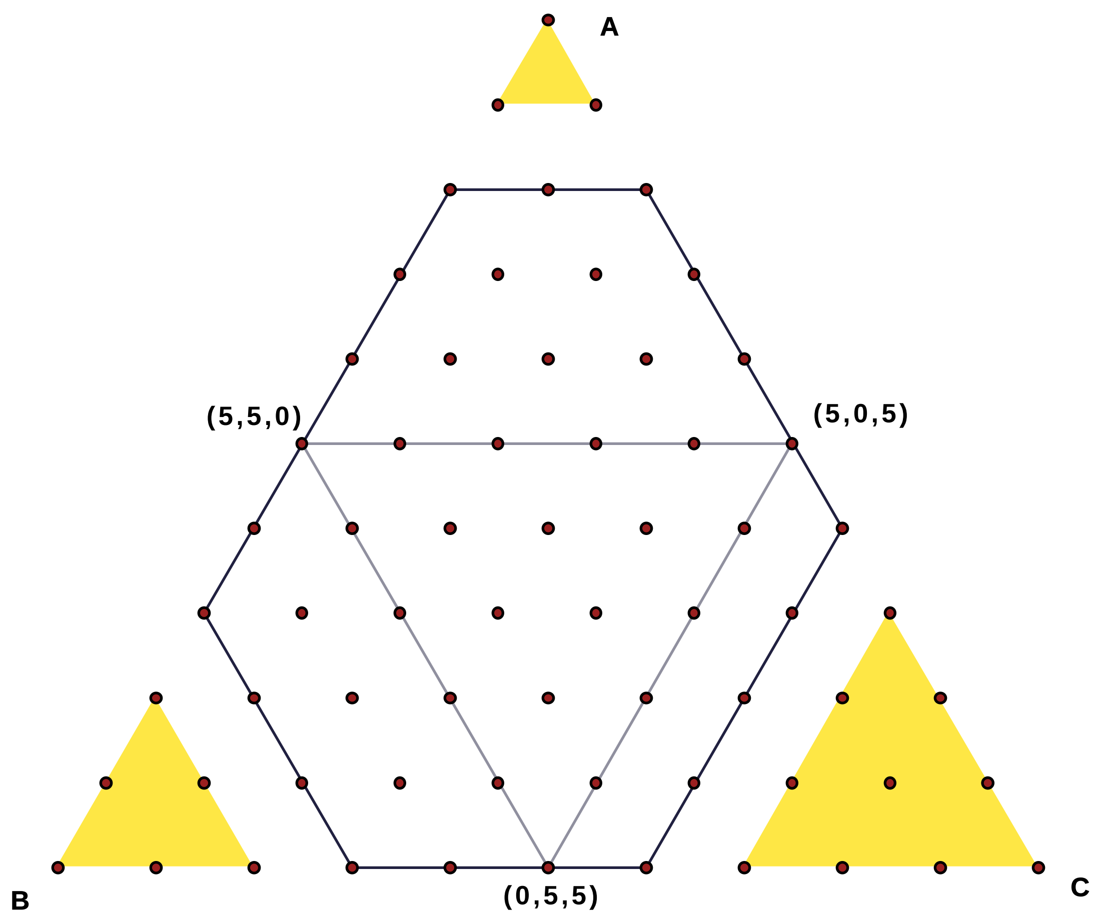
Solution Path



An Interesting Example with No Solution: [10, (8,7,6)]

Can you figure out the notation? A total of 10 points with capacity of A, B, C being 8, 7, 6 respectively.

Now start with any distribution and try to achieve 5 pints.
It turns out that this is impossible unless you start with a 5, for the 3 points (5,5,0), (0,5,5), and (5,0,5) make a closed triangular pouring cycle. They only pour into each other and no other state can pour to get into them. See the figure. What are the other points on the dashed lines forming the sides of the pouring cycle triangle?



Other Problems

Problem [8, (7,5,3)]

Try this problem, with a total of 8 pints.

- Jug A has capacity 7 pints.
- Jug B has capacity 5 pints.
- Jug C has capacity 3 pints.

Find a solution with 4 pints in one jug, starting with distribution (5, 3, 0)

Make up your own problem and solve it!

Extensions and Connections: Trilinear Coordinates and Barycentric Coordinates

If you want to look up geometry connections in reference books or the web, probably searching on the word "jug" will not help much.

What we have done in out solution is to locate points in a triangle using 3 numbers whose sum is constant. For example we label points with (a, b, c), where a+b+c = 8.
There is really no reason to stick with integer a, b, c. If you choose any such a, b, c you can locate points for those values as well.

Also, you can standardize (normalize) the coordinates by assuming the sum is 1. Just replace the integer (a, b, c) by (a/8, b/8, c/8). Then A is always (1, 0, 0), etc. Notice that the midpoints of the sides are then (1/2, 1/2, 0), (1/2, 0, 1/2), (0, 1/2, 1/2), and the centroid (center of mass of A, B, C) is (1/3, 1/3, 1/3).

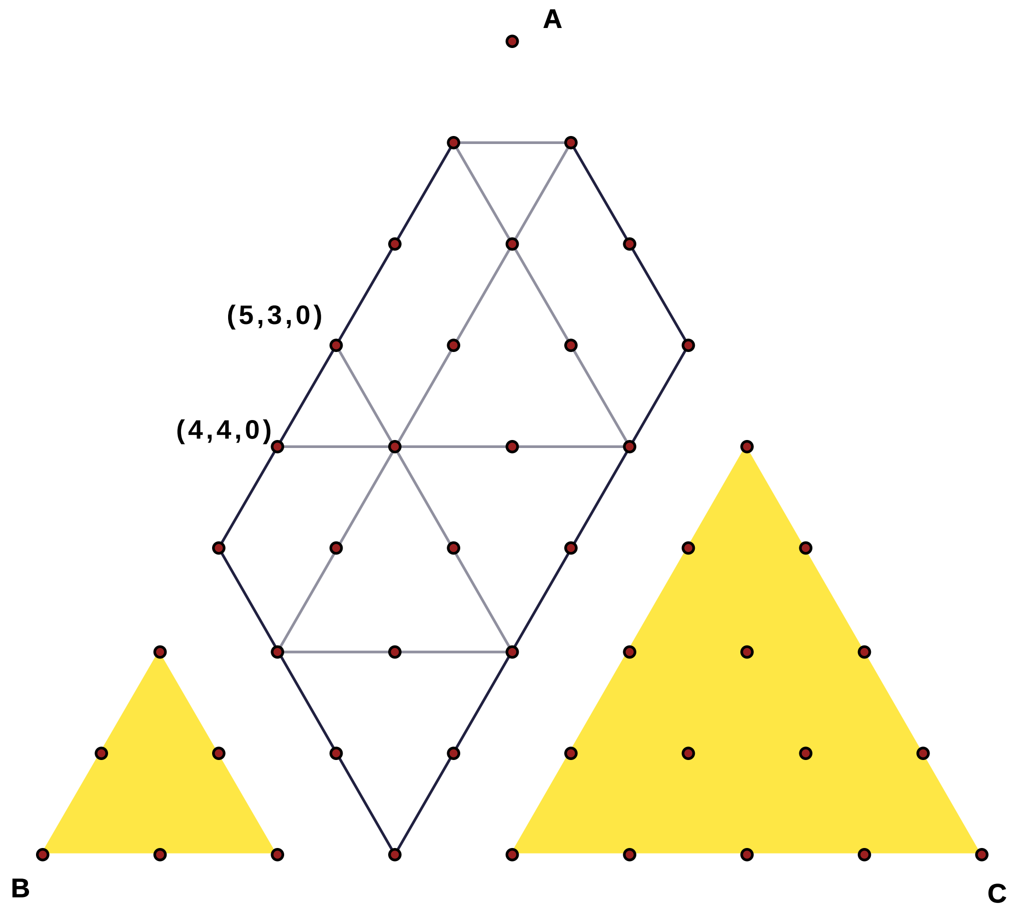
These coordinates are called **barycentric coordinates**. For an equilateral triangle the numbers are also proportional to the distances of the point to the 3 sides. Such distance coordinates are called **trilinear coordinates**. (Things work for general triangles also, but the relationship is a bit more complicated.)

References:

This presentation takes most of its ideas and some of its examples from

- *Geometry Revisited*, H. S. M. Coxeter and S. L. Greitzer, Mathematical Association of American (www.maa.org).

Solution to Problem [8, (7,5,3)]



Triangular Dots that can be used for Problems