Lecturer: Thomas Rothvoss Due date: Friday, Dec 9, 2022, 11pm, on GradeScope

Problem Set 9

514 - Networks and Combinatorial Optimization

Autumn 2022

Exercise 10.22 (10pts)
Derive König's theorem[1] from Edmonds' matroid intersection theorem.

Exercise 10.26 (10pts)
Prove the following Theorem of Rado from 1942: Let $M = (X, \mathcal{I})$ be a matroid and let $X_1, \dots, X_m \subseteq X$. Then there exists a transversal $S \in \mathcal{I}$ for $X_1, \dots, X_m$, if and only if for any $J \subseteq [m]$ one has $r_M(\bigcup_{i \in J} X_i) \geq |J|$.

Hint: Recall that a *transversal* for sets $X_1, \dots, X_m \subseteq X$ is a set $S \subseteq X$ such that there is an bijective map $f : S \to [m]$ with $x \in X_{f(x)}$ for $x \in S$. For any sets $X_1, \dots, X_m \subseteq X$, the pair $M' = (X, \mathcal{I}')$ with $\mathcal{I}' := \{S \subseteq X \mid S \text{ is contained in a transversal of } X_1, \dots, X_m\}$ is a matroid, called the *transversal matroid*. Moreover the rank function of that matroid is

$$r_{M'}(T) = \min_{J \subseteq [m]} \left\{ \left| T \cap \bigcup_{i \in J} X_i \right| + m - |J| \right\}$$

You may use these properties without proof.

Remark. Both exercises are taken from A. Schrijver's lecture notes.

[1] In any bipartite graph $G = (V, E)$ one has $\nu(G) = \tau(G)$.