Lecturer: Thomas Rothvoss   Due date: Friday, Nov 17, 2023, 11pm, on GradeScope

Problem Set 7

514 - Networks and Combinatorial Optimization

Autumn 2023

Exercise 7.1 (10pts)
Let $G = (V,E)$ be a 3-regular graph without any bridge. Show that G has a perfect matching. (A *bridge* is an edge e not contained in any circuit; equivalently, deleting e increases the number of components; equivalently $\{e\}$ is a cut. A graph G is *k-regular* if all vertices have degree k.)

Exercise 7.2 (10pts)
Let $G = (V,E)$ be a graph and let $T \subseteq V$. Then G has a matching covering T if and only if the number of odd components of $G \setminus W$ contained in T is at most $|W|$, for each $W \subseteq V$.

Exercise 7.3 (10pts)
Recall that for a graph $G = (V,E)$ we have defined $P_{\text{matching}}(G) = \text{conv}\{\chi^M \in \mathbb{R}^E \mid M \subseteq E \text{ is a matching}\}$. Prove that for any $k \in \mathbb{N}$,

$$P_{\text{matching}}(G) \cap \{x \in \mathbb{R}^E \mid \mathbf{1}^T x = k\} = \text{conv}\{\chi^M \mid M \subseteq E \text{ is a matching with } |M| = k\}$$

Hint: Let $\mathcal{F} := \{M \subseteq E \mid M \text{ is a matching}\}$. For the non-trivial direction, take a vector $x^* \in P_{\text{matching}}(G) \cap \{x \in \mathbb{R}^E \mid \mathbf{1}^T x = k\}$ and consider the vector $\lambda \in \mathbb{R}^{\mathcal{F}}_{\geq 0}$ that minimizes the function $G(\lambda) := \sum_{M \in \mathcal{F}} \lambda_M \cdot \big||M| - k\big|$ subject to $\sum_{M \in \mathcal{F}} \lambda_M = 1$ and $x^* = \sum_{M \in \mathcal{F}} \lambda_M \chi^M$.

Remark. All three exercises are taken from A. Schrijver's lecture notes.