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MINIMAX THEOREMS AND CONJUGATE SADDLE-FUNCTIONS

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1. Introduction.

The relative interior of a convex set A in R^m, which we denote by $\mathrm{ri}\, A$, is the interior of A with respect to the smallest affine manifold containing it. We denote the closure of A by $\mathrm{cl}\, A$. It is well known that $\mathrm{cl}(\mathrm{ri}\, A) = \mathrm{cl}\, A$ and $\mathrm{ri}(\mathrm{cl}\, A) = \mathrm{ri}\, A$, and in particular that $\mathrm{ri}\, A \neq \emptyset$ when $A \neq \emptyset$.

By a *saddle-element* on $R^m \times R^n$ we shall mean a triple $\{A, B, K\}$, where $A \subseteqq R^m$ and $B \subseteqq R^n$ are non-empty convex sets, and K is a real-valued function on $A \times B$, such that $K(x,y)$ is convex on B for each $x \in A$ and concave on A for each $y \in B$. If A and B are relatively open, we say $\{A, B, K\}$ is *relatively open*. We say $\{A, B, K\}$ is *closed* (resp. *completely closed*) if the pair $\{B, K(x, \cdot)\}$ is a closed convex function in the sense of Fenchel [1] for every $x \in \mathrm{ri}\, A$ (resp. $x \in A$), and $\{A, K(\cdot, x)\}$ is a closed concave function for each $y \in \mathrm{ri}\, B$ (resp. $y \in B$). We say $\{A', B', K'\}$ is *equivalent* to $\{A, B, K\}$ if $A' = A$, $B' = B$ and K' agrees with K on $A \times B$ except perhaps at "corner points", i.e. points $\langle x, y\rangle \in A \times B$ such that $x \notin \mathrm{ri}\, A$ and $y \notin \mathrm{ri}\, B$.

The saddle-elements studied in minimax theory have almost always been ones with A and B closed, and $K(x,y)$ upper semi-continuous in x and lower semi-continuous in y. Every such saddle-element is completely closed (but not conversely). The best known minimax theorems deal only with the case where A and B are actually compact. For the sake of applications to convex programming, however, work has also been done on the non-compact case (e.g. see [4], [5], [6]). In this paper we shall study saddle-elements which might not even be completely closed, but merely closed. We hope to convince the reader that, in many ways, this is really the natural category for minimax theory.

Being "closed" is, as we shall prove, a constructive property in the

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following sense. *Each relatively open saddle-element determines a unique equivalence class of closed saddle-elements, and conversely.*

A completely closed saddle-element is necessarily the sole member of its equivalence class. The converse is not true. As an example, let $R^m = R = R^n$, $A = [0, \infty[$, $B =]0, \infty[$, and $K(x, y) = x/y$ on $A \times B$. Then $\{A, B, K\}$ is the sole member of its equivalence class, but $\{B, K(0, \cdot)\}$ is not a closed convex function. This example also shows that $\{A, B, K\}$ can be closed without $A \times B$ being closed. An interesting and typical example of a non-trivial equivalence class on $R \times R$ is the set of closed saddle-elements

$$\{[0,1],\ [0,1],\ K_\lambda\}, \quad 0 \leqq \lambda \leqq 1\ ,$$

where $K_\lambda(x, y) = x^y$ except when $x = 0 = y$, $K_\lambda(0, 0) = \lambda$.

Let $\{A, B, K\}$ be any closed saddle-element on $R^m \times R^n$, and consider the functions $\underline{L}$ and $\bar{L}$ defined on all of $R^m \times R^n$ by

$$(1.1) \qquad \begin{aligned} \bar{L}(u, v) &= \inf_{x \in A} \sup_{y \in B} \{(x, u) + (y, v) - K(x, y)\}\ , \\ \underline{L}(u, v) &= \sup_{y \in B} \inf_{x \in A} \{(x, u) + (y, v) - K(x, y)\}\ , \end{aligned}$$

where $(.,.)$ denotes inner product. We shall see that $\underline{L}$ and $\bar{L}$ depend only on the equivalence class of $\{A, B, K\}$, and that the set of $\langle u, v\rangle$ where $\underline{L}$ and $\bar{L}$ are both finite is of the form $C \times D$, where C and D are non-empty convex sets in R^m and R^n, respectively. Furthermore, it will be proved that $\{C, D, \underline{L}\}$ and $\{C, D, \bar{L}\}$ are equivalent closed saddle-elements on $R^m \times R^n$. Any saddle-element in this equivalence class will be called a *conjugate* of $\{A, B, K\}$. It turns out that $\{C, D, L\}$ is conjugate to $\{A, B, K\}$ if and only if $\{A, B, K\}$ is conjugate to $\{C, D, L\}$. *Thus the conjugate relation among closed saddle-elements is symmetric, and one-to-one up to equivalence.*

A closed convex function $\{B, f\}$ on R^n can always be viewed as a closed saddle-element on $R^0 \times R^n$, where R^0 is the degenerate zero-dimensional vector space. In this case, the conjugate relation defined above reduces to the one discovered by Fenchel [1].

The notion of conjugacy has obvious applications to minimax theory. If the closed saddle-elements $\{A, B, K\}$ and $\{C, D, L\}$ are conjugate to one another, we have

$$(1.2) \qquad -L(0, 0) = \sup_{x \in A} \inf_{y \in B} K(x, y) = \inf_{y \in B} \sup_{x \in A} K(x, y)$$

if $0 \in \operatorname{ri} C$ and $0 \in D$, or if $0 \in C$ and $0 \in \operatorname{ri} D$, because the conjugates of $\{A, B, K\}$ are all equivalent. We shall see that, if both $0 \in \operatorname{ri} C$ and $0 \in \operatorname{ri} D$, then the minimax in (1.2) is actually attained at a saddle-point.

Our proof of this fact depends on first characterizing saddle-points using the concept of a subgradient of a convex function. Notice in particular that $C = R^m$ and $D = R^n$ by their definition, when A and B are compact and K is continuous on $A \times B$. Hence the minimax results just described include Kakutani's extension [3] of the von Neumann minimax theorem. In the general case as well, it will be shown that C and D can be characterized directly in terms of $\{A, B, K\}$. Thus the minimax results can be applied without having to calculate $\underline{L}$ and $\bar{L}$ (which would beg the question).

In proving the facts we have outlined, it is useful to translate everything about saddle-elements into the context of what we call "saddle-functions" on $R^m \times R^n$. These are everywhere-defined and possibly infinite-valued, but are still concave-convex in a natural sense. Such functions have also been put to good use in minimax theory by Moreau [4]. For the most part, the switch to saddle-functions is a matter of notational convenience. Properties of saddle-functions have to be studied in detail, in all events, because such functions arise in (1.1) as $\underline{L}$ and $\bar{L}$. Since theorems about saddle-elements turn out to be easy consequences of theorems about saddle-functions, it is simpler to concentrate almost entirely on the latter. The saddle-function theory has other advantages of its own. For instance, it enables us to answer questions about the cases in (1.1) where $\underline{L}$ and $\bar{L}$ are not both finite.

2. Convex functions with infinite values.

A *convex function* on R^n is an everywhere-defined function f with values $-\infty \leqq f(x) \leqq +\infty$, such that

$$\{\langle y, \mu\rangle \mid y \in R^n, \mu \in R, f(y) \leqq \mu\} \tag{2.1}$$

is a convex set in R^{n+1}. This condition is satisfied if and only if the inequality

$$f(\lambda y_1 + (1-\lambda) y_2) \leqq \lambda f(y_1) + (1-\lambda) f(y_2) \qquad \text{for } 0 < \lambda < 1 \tag{2.2}$$

holds whenever $f(y_1) < +\infty$ and $f(y_2) < +\infty$. The convex set

$$\operatorname{dom} f = \{y \mid f(y) < +\infty\}$$

is called the *effective domain* of f. If $f(y) > -\infty$ for all y, and $f(y) < +\infty$ for at least one y, we say f is *proper*.

Given a finite-valued convex function on a non-empty convex set $C \subseteqq R^n$, one can always extend it to be $+\infty$ outside of C, and obtain in this way a proper convex function on R^n whose effective domain is C.

Thus the pairs $\{C,f\}$ which are convex functions in the sense of Fenchel correspond one-to-one with the proper convex functions on R^n in the present sense.

The following results were all proved by Fenchel in [1] and [2] (except for trivial extensions to the improper case). We are summarizing them here in our different notation for convenience in later sections.

The *closure* of a convex function f on R^n is the convex function $\mathrm{cl}f$ on R^n which is the supremum of all the affine functions $h \leqq f$ (with the constant functions $-\infty$ and $+\infty$ treated as affine). Obviously

$$(2.3) \qquad \mathrm{cl}f \leqq f, \qquad \mathrm{cl}(\mathrm{cl}f)) = \mathrm{cl}f, \qquad \text{and} \qquad \mathrm{cl}f_1 \leqq \mathrm{cl}f_2 \text{ if } f_1 \leqq f_2 .$$

If $\mathrm{cl}f=f$, we say f is closed. From the definition, one has

$$(2.4) \qquad (\mathrm{cl}f)(y) = \sup_v \inf_z \{(y-z,v)+f(z)\} \qquad \text{for all } y .$$

It is also known that the formulas

$$(2.5) \qquad (\mathrm{cl}f)(y) = \liminf_{z\to y} f(z)$$

$$(2.6) \qquad (\mathrm{cl}f)(y) = \lim_{\lambda\downarrow 0} f(\lambda\bar{y}+(1-\lambda)y) \qquad \text{for any } \bar{y}\in \mathrm{ri}(\mathrm{dom}f) ,$$

are valid whenever $f(z) > -\infty$ for all z, or whenever $y \in \mathrm{cl}(\mathrm{dom}f)$. In particular, a proper convex function is closed if and only if it is everywhere lower semi-continuous. One always has

$$(2.7) \qquad (\mathrm{cl}f)(y) = f(y) \qquad \text{for } y\in \mathrm{ri}(\mathrm{dom}f) .$$

On the other hand,

$$(2.8) \qquad (\mathrm{cl}f)(y) = f(y) = +\infty \qquad \text{for } y \notin \mathrm{cl}(\mathrm{dom}f)$$

provided f does not have the value $-\infty$. If $f(y) = -\infty$ for some y, then $(\mathrm{cl}f)(y) = -\infty$ for all y. If f is identically $+\infty$, then so is $\mathrm{cl}f$. Thus the only improper closed convex functions are the constants $-\infty$ and $+\infty$. If f is proper, then $\mathrm{cl}f$ is proper and coincides with f except perhaps at relative boundary points of $\mathrm{dom}f$, as (2.7) and (2.8) indicate. The values of $\mathrm{cl}f$ at such relative boundary points can be found from (2.6) making use only of the values of f on $\mathrm{ri}(\mathrm{dom}f)$.

A vector $\bar{v}\in R^n$ is said to be a *subgradient* of a convex function f at a point $\bar{y}$ if

$$(2.9) \qquad f(y) \geqq f(\bar{y})+(y-\bar{y},\bar{v}) \qquad \text{for all } y \in R^n .$$

The set of subgradients of f at $\bar{y}$ is a closed convex set which we denote by $\partial f(\bar{y})$. If f is finite and differentiable at $\bar{y}$, then $\partial f(\bar{y})$ contains exactly one vector, which is the ordinary gradient $\nabla f(\bar{y})$. In general,

$$(2.10) \qquad (\partial f)(\bar{y}) \neq \emptyset \quad \text{if } \bar{y} \notin \operatorname{ri}(\operatorname{dom} f) .$$

The infimum of f on R^n is attained at $\bar{y}$ if and only if $0 \in \partial f(\bar{y})$. If f is not identically $+\infty$, one automatically has $\bar{y} \in \operatorname{dom} f$ in this case. Notice from the formulas of the last paragraph that

$$(2.11) \qquad \inf_y f(y) = \inf_y (\operatorname{cl} f)(y) = \inf \{f(y) \mid y \in \operatorname{ri}(\operatorname{dom} f)\} .$$

For *concave functions* on R^n, the facts and definitions which will be needed are obtained from those above by interchanging $\leqq$ with $\geqq$, $+\infty$ with $-\infty$, and infimum with supremum, whereever these occur.

3. Saddle-functions with infinite values.

A *saddle-function* on $R^m \times R^n$ is an everywhere-defined function K with values $-\infty \leqq K(x,y) \leqq +\infty$, such that K is convex in y for each x, and concave in x for each y. It is always true that

$$(3.1) \qquad \sup_x \inf_y K(x,y) \leqq \inf_y \sup_x K(x,y) .$$

The two quantities in (3.1) will be called the *lower* and *upper saddle-values* of K, respectively. When they are equal, we speak simply of the *saddle-value*. A pair $\langle \bar{x}, \bar{y} \rangle$ in $R^m \times R^n$ is called a *saddle-point* of K if

$$(3.2) \qquad K(x,\bar{y}) \leqq K(\bar{x},\bar{y}) \leqq K(\bar{x},y) \qquad \text{for all } x \text{ and } y .$$

If such a saddle-point exists, then K has the saddle-value $K(\bar{x},\bar{y})$.

We shall say that a saddle-function K_2 is a *minimax extension* of a saddle-function K_1 if

$$(3.3) \qquad \begin{aligned} \sup_x \{K_2(x,y) - (x,u)\} &\leqq \sup_x \{K_1(x,y) - (x,u)\} \qquad \text{for all } u \text{ and } y , \\ \inf_y \{K_2(x,y) - (y,v)\} &\geqq \inf_y \{K_1(x,y) - (y,v)\} \qquad \text{for all } x \text{ and } v . \end{aligned}$$

This implies in particular that

$$\sup_x \inf_y K_1 \leqq \sup_x \inf_y K_2 \leqq \inf_y \sup_x K_2 \leqq \inf_y \sup_x K_1 ,$$

and that every saddle-point of K_1 is a saddle-point of K_2. The minimax extension relation is obviously a weak partial ordering of the set of all saddle-functions on $R^m \times R^n$. If K_1 and K_2 are minimax extensions of each other, we say they are *minimax equivalent*. Then, for each $u \in R^m$ and $v \in R^n$, the saddle-functions

$$K_1(x,y) - (x,u) - (y,v) \qquad \text{and} \qquad K_2(x,y) - (x,u) - (y,v)$$

have the same upper and lower saddle-values and the same saddle-points. A saddle-function will be called *closed* if it is minimax equivalent to all its minimax extensions. The definitions suggest that such saddle-functions are more likely to have saddle-values and saddle-points than arbitrary ones, and thus will be better to work with when developing minimax theorems. In the next section we shall study the existence and properties of closed minimax extensions. Some basic facts, needed for this purpose, will be proved here.

For each saddle-function K we define

$$\begin{aligned} \operatorname{dom}_1 K &= \{x \mid K(x,y) > -\infty \text{ for all } y\}, \\ \operatorname{dom}_2 K &= \{y \mid K(x,y) < +\infty \text{ for all } x\}, \\ \operatorname{dom}_1' K &= \{x \mid K(x,y) > -\infty \text{ for all } y \in \operatorname{dom}_2 K\}, \\ \operatorname{dom}_2' K &= \{y \mid K(x,y) < +\infty \text{ for all } x \in \operatorname{dom}_1 K\}. \end{aligned} \tag{3.4}$$

Trivially, $\operatorname{dom}_1 K \subseteqq \operatorname{dom}_1' K$ and $\operatorname{dom}_2 K \subseteqq \operatorname{dom}_2' K$.

LEMMA 1. *The four sets in* (3.4) *are convex. For each* $x \in \operatorname{ri}(\operatorname{dom}_1 K)$, *the effective domain of the convex function* $K(x,\cdot)$ *is* $\operatorname{dom}_2' K$. *For each* $y \in \operatorname{ri}(\operatorname{dom}_2 K)$, *the effective domain of the concave function* $K(\cdot,y)$ *is* $\operatorname{dom}_1' K$.

PROOF. By definition, $\operatorname{dom}_2 K$ is the intersection of the (convex) effective domains of the various functions $K(x,\cdot)$ as x ranges over R^m, and hence it is convex. The convexity of the other three sets follows likewise. Now suppose that $x = \lambda x_1 + (1-\lambda)x_2$ where $0 < \lambda < 1$ and $x_2 \in \operatorname{dom}_1 K$. Then $K(x_2,y) > -\infty$ for all y, so, by the concavity of K in the first argument,

$$K(x,y) \geqq \lambda K(x_1,y) + (1-\lambda)K(x_2,y) = +\infty$$

whenever $K(x_1,y) = +\infty$. Therefore $\operatorname{dom} K(x,\cdot) \subseteqq \operatorname{dom} K(x_1,\cdot)$. In particular, given any $x \in \operatorname{ri}(\operatorname{dom}_1 K)$ and any x_1 in the smallest affine manifold containing $\operatorname{dom}_1 K$, one can choose an $x_2 \in \operatorname{dom}_1 K$ such that $x = \lambda x_1 + (1-\lambda)x_2$ where $0 < \lambda < 1$. Thus the second assertion of the Lemma is true. The third assertion has a parallel proof.

If $\operatorname{dom}_1 K \neq \emptyset$ and $\operatorname{dom}_2' K \neq \emptyset$, we say K is *lower proper*. The restriction of K to the product of the relative interiors of $\operatorname{dom}_1 K$ and $\operatorname{dom}_2' K$ is then a (finite-valued) relatively open saddle-element on $R^m \times R^n$ which will be called the *lower kernel* of K. Similarly, if $\operatorname{dom}_1' K \neq \emptyset$ and $\operatorname{dom}_2 K \neq \emptyset$ we say K is *upper proper*. The restrictions of K to $\operatorname{ri}(\operatorname{dom}_1' K) \times \operatorname{ri}(\operatorname{dom}_2 K)$ is then the *upper kernel* of K. In dealing with the improper cases, it is convenient to introduce an "empty saddle-element of type $-\infty$ and an empty saddle-element of type $+\infty$". We say that the lower (resp. upper) kernel of K is $-\infty$ if $\operatorname{dom}_1 K = \emptyset$ (resp. if

$\mathrm{dom}_1' K = \emptyset$, but $\mathrm{dom}_2 K \neq \emptyset$); it is $+\infty$ if $\mathrm{dom}_1 K \neq \emptyset$, but $\mathrm{dom}_2' K = \emptyset$ (resp. if $\mathrm{dom}_2 K = \emptyset$). This terminology will be justified later.

A saddle-function K will be called *simple* if its lower and upper kernels are the same. Then one can speak of the *kernel* of K, or of K being *proper*, without having to distinguish between "lower" and "upper". According to the definitions, the saddle-functions which are simple and proper are the ones such that

$$\text{(3.5)} \quad \mathrm{dom}_1' K \subseteq \mathrm{cl}(\mathrm{dom}_1 K) \neq \emptyset \quad \text{and} \quad \mathrm{dom}_2' K \subseteq \mathrm{cl}(\mathrm{dom}_2 K) \neq \emptyset .$$

The kernel of K is then, of course, the same as the restriction of K to the relative interior of $(\mathrm{dom}_1 K) \times (\mathrm{dom}_2 K)$.

Not every saddle-function is simple. For example, let $R^m = R^n$ and define $K(x,y) = +\infty$ when $(x,y) > 0$, $K(x,y) = 0$ when $(x,y) = 0$, $K(x,y) = -\infty$ when $(x,y) < 0$. Then K is both lower and upper proper, but not simple. The proof of Lemma 1 actually shows, however, that $\mathrm{dom}_1' K = \mathrm{dom}_1 K$ and $\mathrm{dom}_2' K = \mathrm{dom}_2 K$ whenever $\mathrm{dom}_1 K$ and $\mathrm{dom}_2 K$ both have non-empty interiors. Thus K is always simple and proper in this case. Non-simple saddle-functions are therefore rather freakish.

Given any (non-empty) saddle-element $\{A, B, K\}$ on $R^m \times R^n$, we can always set

$$\text{(3.6)} \qquad K(x,y) = \begin{cases} +\infty & \text{if } x \in A \text{ and } y \notin B , \\ -\infty & \text{if } x \notin A . \end{cases}$$

Then K becomes a simple proper saddle-function, with

$$\mathrm{dom}_1 K = A = \mathrm{dom}_1' K, \qquad \mathrm{dom}_2 K = B = \mathrm{dom}_2' K .$$

The upper and lower saddle-values and saddle-points of this saddle-function are the same as those of $\{A, B, K\}$ in the ordinary sense. This would also be true if, instead of (3.6), one sets $K(x,y) = -\infty$ if $x \notin A$ and $y \in B$, $K(x,y) = +\infty$ if $y \notin B$. The second saddle-function is minimax equivalent to the first one.

For each saddle-function K on $R^m \times R^n$, we denote by $\mathrm{cl}_1 K$ the function on $R^m \times R^n$ obtained by closing $K(x,y)$ as a concave function of x, for each y. Similarly, $\mathrm{cl}_2 K$ denotes the function obtained by closing $K(x,y)$ as a convex function of y, for each x. Obviously

$$(\mathrm{cl}_2 K)(x,y) \leqq K(x,y) \leqq (\mathrm{cl}_1 K)(x,y)$$

for all x and y.

LEMMA 2. *Let K be any saddle-function. Then $\mathrm{cl}_1 K$ and $\mathrm{cl}_2 K$ are simple saddle-functions, and both are minimax extensions of K. The kernel of $\mathrm{cl}_1 K$ is the upper kernel of K, and the kernel of $\mathrm{cl}_2 K$ is the lower kernel of K.*

PROOF. For each x, $(\mathrm{cl}_2 K)(x,\cdot)$ is the closure of the convex function $K(x,\cdot)$, and hence is convex. From the formulas in § 2 we have

$$(3.7)\qquad \begin{aligned} &(\mathrm{cl}_2 K)(x,y) = -\infty && \text{if } x \notin \mathrm{dom}_1 K\,, \\ &(\mathrm{cl}_2 K)(x,y) = \liminf_{z\to y} K(x,z) > -\infty && \text{if } x \in \mathrm{dom}_1 K\,. \end{aligned}$$

To show now that $\mathrm{cl}_2 K$ is concave in its first argument, fix any y and choose any x_1, x_2, λ, with

$$\mathrm{cl}_2 K(x_1,y) > -\infty, \quad \mathrm{cl}_2 K(x_2,y) > -\infty, \quad 0<\lambda<1\,.$$

Then x_1 and x_2 belong to $\mathrm{dom}_1 K$ by (3.7), and hence so does $x=\lambda x_1+(1-\lambda)x_2$. The functions $K(x_1,\cdot)$, $K(x_2,\cdot)$ and $K(x,\cdot)$ thus never have the value $-\infty$. It follows from (3.7) and the concavity of K in its first argument that

$$\begin{aligned} (\mathrm{cl}_2 K)(x,y) &\geqq \liminf_{z\to y}\,[\lambda K(x_1,z)+(1-\lambda)K(x_2,z)] \\ &\geqq \lambda \liminf_{z_1\to y} K(x_1,z_1)+(1-\lambda)\liminf_{z_2\to y} K(x_2,z_2) \\ &= \lambda(\mathrm{cl}_2 K)(x_1,y)+(1-\lambda)(\mathrm{cl}_2 K)(x_2,y)\,. \end{aligned}$$

Therefore $\mathrm{cl}_2 K$ is a saddle-function. If K is lower proper, then

$$(3.8)\qquad \begin{aligned} &\mathrm{dom}_1 K = \mathrm{dom}_1 \mathrm{cl}_2 K = \mathrm{dom}_1{}' \mathrm{cl}_2 K\,, \\ &\mathrm{dom}_2{}' K \subseteqq \mathrm{dom}_2 \mathrm{cl}_2 K = \mathrm{dom}_2{}' \mathrm{cl}_2 K \subseteqq \mathrm{cl}(\mathrm{dom}_2{}' K)\,, \end{aligned}$$

by (3.7) and Lemma 1. In this case $\mathrm{cl}_2 K$ is therefore simple and proper. Furthermore, $(\mathrm{cl}_2 K)(x,y)=K(x,y)$ by Lemma 1 and formula (2.7) when $x\in \mathrm{ri}(\mathrm{dom}_1 K)$ and $y\in\mathrm{ri}(\mathrm{dom}_2{}' K)$. Thus the kernel of $\mathrm{cl}_2 K$ is the lower kernel of K in this case. If $\mathrm{dom}_1 K \neq \emptyset$ but $\mathrm{dom}_2{}' K=\emptyset$, (3.8) still holds if $\mathrm{dom}_1{}'\mathrm{cl}_2 K$ is omitted. Then K and $\mathrm{cl}_2 K$ are both simple saddle-functions with kernel $+\infty$. If $\mathrm{dom}_1 K=\emptyset$, that is, the lower kernel of K is $-\infty$, then $(\mathrm{cl}_2 K)(x,y)=-\infty$ for all x and y. Then $\mathrm{cl}_2 K$ is simple and has kernel $-\infty$. Finally, we must verify that $\mathrm{cl}_2 K$ is a minimax extension of K. The first inequality in (3.3) is trivially satisfied for $K_2=\mathrm{cl}_2 K$ and $K_1=K$, because $\mathrm{cl}_2 K\leqq K$. The two sides of the second inequality are actually equal in all cases, in view of (3.7).

LEMMA 3. *If K_2 is a minimax extension of K_1, then* $\mathrm{cl}_1 K_2 \leqq \mathrm{cl}_1 K_1$ *and* $\mathrm{cl}_2 K_2 \geqq \mathrm{cl}_2 K_1$.

PROOF. The second inequality follows from the second inequality in (3.3) via formula (2.4) for the closure of a convex function. The first inequality is its concave counterpart.

LEMMA 4. *Let K be a simple saddle-function on $R^m \times R^n$, and let $\overline{K} = \mathrm{cl}_1\mathrm{cl}_2 K$ and $\underline{K} = \mathrm{cl}_2\mathrm{cl}_1 K$. If K has the kernel $-\infty$, then $\underline{K} = \overline{K} \equiv -\infty$. If K has the kernel $+\infty$, then $\underline{K} = \overline{K} \equiv +\infty$. If K is proper, then the values of $\underline{K}$ and $\overline{K}$ depend only on the kernel of K, and there exists a pair of non-empty convex sets $C \subseteq R^m$ and $D \subseteq R^n$ such that $\underline{K}(x,y)$ and $\overline{K}(x,y)$ satisfy the relations in Table 1 for various locations of $\langle x,y\rangle$ with respect to $C \times D$.*

	$\mathrm{ri}\, D$	$D \setminus \mathrm{ri}\, D$	$\mathrm{cl}\, D \setminus D$	$R^n \setminus \mathrm{cl}\, D$
$\mathrm{ri}\, C$	$-\infty < \underline{K} = \overline{K} < \infty$			$-\infty < \underline{K} = \overline{K} = \infty$
$C \setminus \mathrm{ri}\, C$		$-\infty < \underline{K} \leqq \overline{K} < \infty$	$-\infty < \underline{K} \leqq \overline{K} = \infty$	
$\mathrm{cl}\, C \setminus C$		$-\infty = \underline{K} \leqq \overline{K} < \infty$		
$R^m \setminus \mathrm{cl}\, C$	$-\infty = \underline{K} = \overline{K} < \infty$			$-\infty = \underline{K} < \overline{K} = \infty$

Table 1.

PROOF. Suppose that K is proper, and choose any $\bar{x} \in \mathrm{ri}\,(\mathrm{dom}_1 K)$ and $\bar{y} \in \mathrm{ri}\,(\mathrm{dom}_2 K)$. Let C be the effective domain of the closure of the proper concave function $K(\cdot, \bar{y})$, and let D be the effective domain of the closure of $K(\bar{x}, \cdot)$. By the concave analog of formula (2.5), the effective domain of $K(\cdot, \bar{y})$ lies between $\mathrm{ri}\, C$ and C. Therefore

$$\bar{x} \in \mathrm{ri}\, C \subseteq \mathrm{dom}_1 K = \mathrm{dom}_1 \mathrm{cl}_2 K \subseteq C \tag{3.9}$$

by Lemma 1 and the first half of (3.8). Since K is simple, it follows now from Lemma 1 and the second half of (3.8) that

$$\bar{y} \in \mathrm{ri}\, D \subseteq \mathrm{dom}_2 K \subseteq \mathrm{dom}_2 \mathrm{cl}_2 K = D\,. \tag{3.10}$$

Furthermore, the relative interior of the effective domain of the concave function $(\mathrm{cl}_2 K)(\cdot, y)$ is $\mathrm{ri}\, C$ for every y by (3.7) and (3.9). The closure formulas for concave functions, along with (3.9) and (3.10), therefore yield

$$\overline{K}(x,y) = \begin{cases} +\infty & \text{if } y \notin D\,, \\ -\infty & \text{if } x \notin \mathrm{cl}\, C,\ y \in D\,, \end{cases} \tag{3.11}$$

$$\overline{K}(x,y) < +\infty \qquad \text{if } y \in D\,,$$

$$\overline{K}(x,y) = \begin{cases} (\mathrm{cl}_2 K)(x,y) & \text{if } x \in \mathrm{ri}\, C\,, \\ \lim\limits_{\lambda \downarrow 0} (\mathrm{cl}_2 K)\big(\lambda\bar{x} + (1-\lambda)x, y\big) & \text{if } x \in \mathrm{cl}\, C\,. \end{cases} \tag{3.12}$$

But $\lambda\bar{x} + (1-\lambda)x$ belongs to $\mathrm{ri}\, C = \mathrm{ri}\,(\mathrm{dom}_1 K)$ for $0 < \lambda < 1$ when $x \in \mathrm{cl}\, C$, so the relative interior of the effective domain of the convex function $K\big(\lambda\bar{x} + (1-\lambda)x, \cdot\big)$ is then $\mathrm{ri}\,(\mathrm{dom}_2 K) = \mathrm{ri}\, D$ by Lemma 1 and (3.10). Applying formulas (2.6) and (2.7) for the closure of this convex function, and substituting into (3.12), we get

$$\overline{K}(x,y) = \begin{cases} K(x,y) & \text{if } x \in \operatorname{ri} C,\ y \in \operatorname{ri} D\,, \\ \lim_{\lambda \downarrow 0} K(\lambda \overline{x} + (1-\lambda)x, y) & \text{if } x \in \operatorname{cl} C,\ y \in \operatorname{ri} D\,, \\ \lim_{\mu \downarrow 0} K(x, \mu \overline{y} + (1-\mu)y) & \text{if } x \in \operatorname{ri} C,\ y \in \operatorname{cl} D\,, \\ \lim_{\lambda \downarrow 0} \lim_{\mu \downarrow 0} K(\lambda \overline{x} + (1-\lambda)x, \mu \overline{y} + (1-\mu)y) & \\ & \text{if } x \in \operatorname{cl} C,\ y \in \operatorname{cl} D\,. \end{cases} \tag{3.13}$$

The second case of (3.13) implies via (3.10) and Lemma 1 that

$$\begin{aligned} \overline{K}(x,y) &= -\infty \quad \text{if } x \notin C,\ y \in \operatorname{ri} D\,, \\ \overline{K}(x,y) &> -\infty \quad \text{if } x \in C\,. \end{aligned} \tag{3.14}$$

If we continue (3.11), (3.13), and (3.14) for $\overline{K}$ together with the dual results for $\underline{K}$, we get all the relations in Table 1 except one: that $\underline{K}(x,y) \leqq \overline{K}(x,y)$ when $x \in C \setminus \operatorname{ri} C$ and $y \in D \setminus \operatorname{ri} D$. The other relations imply, however, that $\operatorname{cl}_2 \underline{K} = \operatorname{cl}_2 \overline{K}$. Since $\operatorname{cl}_2 \underline{K} = \underline{K}$ by the definition of $\underline{K}$, we therefore have $\underline{K} \leqq \overline{K}$. The argument in the cases where K is improper is elementary, as in the proof of Lemma 2.

4. Closed saddle-functions.

The three theorems in this section answer in detail questions about the existence and properties of minimax equivalence classes of closed saddle-functions.

THEOREM 1.

(a) *Each minimax equivalence class E of closed saddle-functions on $R^m \times R^n$ is an "interval" in the following sense. There exist unique saddle-functions $\underline{K} \in E$ and $\overline{K} \in E$, such that a saddle-function K belongs to E if and only if $\underline{K} \leqq K \leqq \overline{K}$. Moreover $\operatorname{cl}_1 K = \overline{K}$ and $\operatorname{cl}_2 K = \underline{K}$ for every $K \in E$.*

(b) *In order that a given pair of saddle-functions $\underline{K}$ and $\overline{K}$ be the unique lower and upper members, respectively, of some minimax equivalence class of closed saddle-functions, it is necessary and sufficient that $\operatorname{cl}_1 \underline{K} = \overline{K}$ and $\operatorname{cl}_2 \overline{K} = \underline{K}$.*

(c) *If $\underline{K}$ and $\overline{K}$ are the unique lower and upper members, respectively, of a minimax equivalence class of closed saddle-functions, then either $\underline{K} = \overline{K} \equiv -\infty$, or $\underline{K} = \overline{K} \equiv +\infty$, or $\underline{K}$ and $\overline{K}$ are both proper and they satisfy the relationships in Table 1 for*

$$C = \operatorname{dom}_1 \underline{K} = \operatorname{dom}_1 \overline{K} \quad \textit{and} \quad D = \operatorname{dom}_2 \underline{K} = \operatorname{dom}_2 \overline{K}\,. \tag{4.1}$$

Proof. Suppose first that $\underline{K}$ and $\overline{K}$ are saddle-functions such that $\mathrm{cl}_1 \underline{K} = \overline{K}$ and $\mathrm{cl}_2 \overline{K} = \underline{K}$. Then each is a minimax extension of the other by Lemma 2, so $\underline{K}$ and $\overline{K}$ belong to the same minimax equivalence class E. By the definition of minimax equivalence, every saddle-function K such that $\underline{K} \leqq K \leqq \overline{K}$ must also belong to E. Assume that K is any minimax extension of $\overline{K}$. Lemma 3 implies

$$\underline{K} = \mathrm{cl}_2 \overline{K} \leqq \mathrm{cl}_2 K \leqq K \leqq \mathrm{cl}_1 K \leqq \mathrm{cl}_1 \overline{K} = \mathrm{cl}_1(\mathrm{cl}_1 \underline{K}) = \overline{K} ,$$

and hence that $K \in E$. Therefore E consists precisely of the saddle-functions K such that $\underline{K} \leqq K \leqq \overline{K}$, and all of these are closed. Conversely, suppose E is a minimax equivalence class of closed saddle-functions and take any $K \in E$. Since K is closed, the functions $\overline{K} = \mathrm{cl}_1 K$ and $\underline{K} = \mathrm{cl}_2 K$ again belong to E. Hence $\mathrm{cl}_1 \underline{K} = \mathrm{cl}_1 \overline{K} = \overline{K}$ and $\mathrm{cl}_2 \overline{K} = \mathrm{cl}_2 \underline{K} = \underline{K}$ by Lemma 3, so the argument in the first part of the proof can be applied to $\underline{K}$ and $\overline{K}$. This proves (a) and (b). Next observe that, if $\underline{K}$ and $\overline{K}$ satisfy the condition in (b), then both are simple saddle-functions by Lemma 2. Furthermore, $\mathrm{cl}_1 \mathrm{cl}_2 \overline{K} = \mathrm{cl}_1 \underline{K} = \overline{K}$ and $\mathrm{cl}_2 \mathrm{cl}_1 \overline{K} = \mathrm{cl}_2 \overline{K} = \underline{K}$. Part (c) is therefore a consequence of Lemma 4.

Corollary 1. *Every closed saddle function is simple. In fact*

$$\mathrm{dom}_1' K = \mathrm{dom}_1 K \qquad \textit{and} \qquad \mathrm{dom}_2' K = \mathrm{dom}_2 K$$

for any closed saddle-function K, and these convex sets depend only on the minimax equivalence class containing K. Moreover, if K is closed and not identically $-\infty$ or $+\infty$, then K is proper and

$$\begin{aligned} \mathrm{sup}_x \mathrm{inf}_y K(x,y) &= \sup\{\inf\{K(x,y) \mid y \in \mathrm{dom}_2 K\} \mid x \in \mathrm{dom}_1 K\} , \\ \mathrm{inf}_y \mathrm{sup}_x K(x,y) &= \inf\{\sup\{K(x,y) \mid x \in \mathrm{dom}_1 K\} \mid y \in \mathrm{dom}_2 K\} . \end{aligned} \tag{4.2}$$

Proof. Let $\underline{K}$ and $\overline{K}$ be the lower and upper members of the minimax equivalence class containing K. Then $\mathrm{dom}_1 K = \mathrm{dom}_1' K = C$ and $\mathrm{dom}_2 K = \mathrm{dom}_2' K = D$ in part (c) of Theorem 1. This verifies all of the corollary except (4.2). The function K' defined by $K'(x,y) = K(x,y)$ for $x \in C$ and $y \in D$, $K'(x,y) = +\infty$ if $x \in C$ and $y \notin D$, $K'(x,y) = -\infty$ if $x \notin C$, also lies between $\underline{K}$ and $\overline{K}$ according to Table 1, and hence it is minimax equivalent to K. The first equation in (4.2) is a consequence of the definition of minimax equivalence, inasmuch as the right side is just the "supinf" of K'. The other equation follows similarly.

Corollary 2. *A saddle-function K is closed if and only if K is minimax equivalent to $\mathrm{cl}_1 K$ and $\mathrm{cl}_2 K$.*

Proof. The necessity of the condition is asserted by part (a) of Theorem 1. On the other hand, if $\mathrm{cl}_1 K$ and $\mathrm{cl}_2 K$ are minimax equivalent, we have

$$\mathrm{cl}_2(\mathrm{cl}_1 K) = \mathrm{cl}_2(\mathrm{cl}_2 K) = \mathrm{cl}_2 K, \qquad \mathrm{cl}_1(\mathrm{cl}_2 K) = \mathrm{cl}_1(\mathrm{cl}_1 K) = \mathrm{cl}_1 K\,,$$

by Lemma 3. Hence $\underline{K} = \mathrm{cl}_2 K$ and $\overline{K} = \mathrm{cl}_1 K$ satisfy the hypothesis of part (b) and consequently are closed. If K is minimax equivalent to $\mathrm{cl}_1 K$ and $\mathrm{cl}_2 K$, it must then be closed as well.

COROLLARY 3. *A saddle-function is closed and proper if and only if it satisfies the following five conditions*:

(a) $\mathrm{dom}_1 K \neq \emptyset$ *and* $\mathrm{dom}_2 K \neq \emptyset$;
(b) $K(x,y) = +\infty$ *when* $x \in \mathrm{dom}_1 K$ *but* $y \notin \mathrm{cl}(\mathrm{dom}_2 K)$;
(c) $K(x,y) = -\infty$ *when* $y \in \mathrm{dom}_2 K$ *but* $x \notin \mathrm{cl}(\mathrm{dom}_1 K)$;
(d) *for each* $x \in \mathrm{ri}(\mathrm{dom}_1 K)$, $K(x,\cdot)$ *is a closed (proper) convex function*;
(e) *for each* $y \in \mathrm{ri}(\mathrm{dom}_2 K)$, $K(\cdot,y)$ *is a closed (proper) concave function.*

PROOF. The necessity of the conditions follows from part (c) of Theorem 1. For the sufficiency, we note first that conditions (a), (c), and (e) imply, via Lemma 1, that $\mathrm{dom}_1 \mathrm{cl}_1 K = \mathrm{dom}_1 K$. Hence

$$\inf_y \{K(x,y) - (y,v)\} = \inf_y \{(\mathrm{cl}_1 K)(x,y) - (y,v)\} \qquad \text{for all } v$$

trivially when $x \notin \mathrm{dom}_1 K$, both sides then being $-\infty$. Furthermore, (b) and (d) imply that for $x \in \mathrm{dom}_1 K$ the convex functions $K(x,\cdot)$ and $(\mathrm{cl}_1 K(x,\cdot)$ both have $\mathrm{ri}(\mathrm{dom}_2 K)$ as the relative interior of their effective domains, and that they agree there. Since the infimum of a convex function f over R^n is the same as the infinum of f on $\mathrm{ri}(\mathrm{dom} f)$ (see (2.11)), the above equation must also hold for $x \in \mathrm{dom}_1 K$. But it is always true, by the concave analog of (2.11), that

$$\sup_x \{K(x,y) - (x,u)\} = \sup_x \{(\mathrm{cl}_1 K)(x,y) - (x,u)\}$$

for all y and u. Therefore $\mathrm{cl}_1 K$ is minimax equivalent to K. By a similar argument, so is $\mathrm{cl}_2 K$. Hence K is closed by Corollary 2, and proper of course by condition (a).

THEOREM 2. *Every saddle-function K has closed minimax extensions. In order that all the closed minimax extensions of K be minimax equivalent, however, it is both necessary and sufficient that K be simple. If K is simple, the lower and upper members of its class of closed minimax extensions are* $\underline{K} = \mathrm{cl}_2 \mathrm{cl}_1 K$ *and* $\overline{K} = \mathrm{cl}_1 \mathrm{cl}_2 K$, *respectively.*

PROOF. If K is simple and proper, the functions $\underline{K} = \mathrm{cl}_1 \mathrm{cl}_2 K$ and $\overline{K} = \mathrm{cl}_1 \mathrm{cl}_2$ satisfy the relationships in Table 1, according to Lemma 4.

It follows from these relationships that $\mathrm{cl}_1 \underline{K} = \mathrm{cl}_1 \overline{K}$ and $\mathrm{cl}_2 \underline{K} = \mathrm{cl}_2 \overline{K}$. The latter is also true trivially by Lemma 4 when K is simple but improper. Since $\mathrm{cl}_1 \overline{K} = \overline{K}$ and $\mathrm{cl}_2 \underline{K} = \underline{K}$ by definition, $\underline{K}$ and $\overline{K}$ must determine according to part (c) of Theorem 1 a minimax equivalence class of closed minimax extensions of K. Assume now that K' is an arbitrary closed minimax extension of K, and let $\underline{K}'$ and $\overline{K}'$ be the lower and upper members of its class. By Lemma 3 and part (a) of Theorem 1 we have $\mathrm{cl}_2 K \leqq \mathrm{cl}_2 \overline{K}' = \underline{K}'$ and

$$\underline{K} = \mathrm{cl}_2 \overline{K} = \mathrm{cl}_2 \mathrm{cl}_1 (\mathrm{cl}_2 K) \leqq \mathrm{cl}_2 \mathrm{cl}_1 \underline{K}' = \mathrm{cl}_2 \overline{K}' = \underline{K}' \leqq K' .$$

Similarly, $K' \leqq \overline{K}$. Thus K belongs to the class determined by $\underline{K}$ and $\overline{K}$. This proves the "if" part of the second statement of the theorem, and the third statement, too. If K is any saddle function, $\mathrm{cl}_2 K$ is a simple minimax extension of K by Lemma 2. Thus $\mathrm{cl}_1 \mathrm{cl}_2 (\mathrm{cl}_2 K) = \mathrm{cl}_1 \mathrm{cl}_2 K$ is a closed minimax extension of $\mathrm{cl}_2 K$, and hence of K, by the part of the theorem we have so far finished proving. Likewise, $\mathrm{cl}_2 \mathrm{cl}_1 K$ is always a closed minimax extension of K. But the kernels of $\mathrm{cl}_1 \mathrm{cl}_2 K$ and $\mathrm{cl}_2 \mathrm{cl}_1 K$ are, from Lemma 2, the lower and upper kernels of K. Closed saddle-functions in the same minimax equivalence class must have the same kernel, according to part (c) of Theorem 1. Hence the closed minimax extensions $\mathrm{cl}_1 \mathrm{cl}_2 K$ and $\mathrm{cl}_2 \mathrm{cl}_1 K$ are not in the same class when K is not simple.

THEOREM 3.

(a) *All the saddle-functions in a minimax equivalence class of closed proper saddle-functions have the same kernel. Conversely each relatively open saddle-element is the kernel for exactly one such class.*

(b) *The set of saddle-elements* $\{\mathrm{dom}_1 K, \mathrm{dom}_2 K, K\}$, *as* K *ranges over a minimax equivalence class of closed proper saddle-functions, forms a (complete) equivalence class of closed saddle-elements. Conversely, every class of the latter sort arises from a unique class of the former sort.*

PROOF. It is immediate from part (c) of Theorem 1 that closed proper saddle-functions K and K' in the same minimax equivalence class have the same kernel, and that

$$\{\mathrm{dom}_1 K, \mathrm{dom}_2 K, K\} \qquad \text{and} \qquad \{\mathrm{dom}_1 K', \mathrm{dom}_2 K', K'\}$$

are equivalent closed saddle-elements. Conversely, given a relatively open saddle-element $\{A, B, K\}$ we can define the values of K outside of $A \times B$ to be those in (3.6). Then K is a simple proper saddle-function. According to Theorem 2, $\mathrm{cl}_1 \mathrm{cl}_2 K$ and $\mathrm{cl}_2 \mathrm{cl}_1 K$ determine the class of closed minimax extensions of K, and their kernel is the same as that of K, in other words it is $\{A, B, K\}$, by Lemma 2. If K' is any other closed saddle-function with this kernel, then $\mathrm{cl}_1 \mathrm{cl}_2 K' = \mathrm{cl}_1 \mathrm{cl}_2 K$, because these functions depend only on the kernel of a simple saddle-function, by Lemma 4. But K', being closed, is minimax equivalent to $\mathrm{cl}_1 \mathrm{cl}_2 K'$. Hence it belongs to the class of closed minimax extensions of K. This proves that only one closed class has kernel $\{A, B, K\}$. To finish the proof of (b), we now take any closed saddle-element $\{A, B, K\}$ and define K outside of $A \times B$ as before. This time K is a closed proper saddle-function in virtue of Corollary 3 to Theorem 1. Of course $\{\mathrm{dom}_1 K, \mathrm{dom}_2 K, K\}$ is just $\{A, B, K\}$. Thus every closed saddle-element arises from a closed proper saddle-function. The completeness and uniqueness in (b) now follow from part (a).

5. Conjugates of closed saddle-functions.

For each saddle-function K on $R^m \times R^n$, the functions $\underline{L}$ and $\bar{L}$ on $R^m \times R^n$ defined by

$$\underline{L}(u,v) = \sup_y \inf_x \{(x,u) + (y,v) - K(x,y)\}, \tag{5.1a}$$

$$\bar{L}(u,v) = \inf_x \sup_y \{(x,u) + (y,v) - K(x,y)\}, \tag{5.1b}$$

will be called the *lower and upper conjugates* of K, respectively.

THEOREM 4. *Let K be any closed saddle-function. Then the lower and upper conjugates $\underline{L}$ and $\bar{L}$ of K are again saddle-functions, and they depend only on the minimax equivalence class containing K. In fact $\underline{L}$ and $\bar{L}$ are the lower and upper members respectively, of a minimax equivalence class of closed saddle-functions. If L is any member of this equivalence class, the lower and upper conjugates $\underline{K}$ and $\bar{K}$ of L are in turn the lower and upper members of the closed minimax equivalence class containing K.*

PROOF. For each v, $\bar{L}(\cdot, u)$ is an infimum of affine functions on $R^m \times R^n$ and hence is a closed concave function. Now fix any $u \in R^m$. Choose any v_1, v_2, μ_1, and μ_2 such that $\bar{L}(u, v_1) \leqq \mu_1 \in R$ and $\bar{L}(u, v_2) \leqq \mu_2 \in R$. To prove the convexity of $\bar{L}(u, \cdot)$ we must show that $\bar{L}(u,v) \leqq \lambda\mu_1 + (1-\lambda)\mu_2$, where $0 < \lambda < 1$ and $v = \lambda v_1 + (1-\lambda) v_2$.

Take an arbitrary $\varepsilon > 0$. By the definition of $\bar{L}$ there exist vectors x_1 and x_2 such that

$$\begin{aligned} \sup_y \{(x_1, u) + (y, v_1) - K(x_1, y)\} &\leqq \mu_1 + \varepsilon, \\ \sup_y \{(x_2, u) + (y, v_2) - K(x_2, y)\} &\leqq \mu_2 + \varepsilon. \end{aligned} \tag{5.2}$$

These inequalities imply that $K(x, y) > -\infty$ and $K(x_2, y) > -\infty$ for all y. Hence, for $x = \lambda x_1 + (1-\lambda) x_2$, the inequality

$$K(x,y) \geqq \lambda K(x_1,y)+(1-\lambda)K(x_2,y)$$

makes sense and is valid for all y by the concavity of K in its first argument. Consequently,

$$\begin{aligned}
&(x,u)+(y,v)-K(x,y) \\
&\quad \leqq \lambda[(x_1,u)+(y,v_1)-K(x_1,y)]+(1-\lambda)[(x_2,u)+(y,v_2)-K(x_2,y)] \\
&\quad \leqq \lambda(\mu_1+\varepsilon)+(1-\lambda)(\mu_2+\varepsilon)
\end{aligned}$$

for all y by (5.2). This shows that, for every $\varepsilon>0$, there exists an x such that

$$\sup_y\{(x,u)+(y,v)-K(x,y)\} \leqq \lambda\mu_1+(1-\lambda)\mu_2+\varepsilon \,. \tag{5.3}$$

The left side of (5.3) is at least as large as $\bar{L}(u,v)$, so this is enough to complete the proof that $\bar{L}$ is a saddle-function. The proof for $\underline{L}$ is parallel. By the definition of minimax equivalence, the values of $\underline{L}$ and $\bar{L}$ depend only on the class containing K. We shall now show that

$$\mathrm{cl}_1\underline{L} = \bar{L} \qquad \text{and} \qquad \mathrm{cl}_2\bar{L} = \underline{L} \,. \tag{5.4}$$

This will verify the second assertion of the theorem, in view of Theorem 1b. Applying formula (2.4) for the closure of a convex function, we get

$$(\mathrm{cl}_2\bar{L})(u,v) = \sup_z\inf_w\{(z,v-w)+\bar{L}(u,w)\} \,. \tag{5.5}$$

If we substitute the formula for $\bar{L}$ into (5.5) and rearrange terms, the right side becomes

$$\begin{aligned}
&= \sup_z\inf_w\inf_x\sup_y\{(z,v-w)+(x,u)+(y,v)-K(x,y)\} \\
&= \sup_z\inf_x\big\{(x,u)+(z,v)-\sup_w\inf_y\{(z-y,w)+K(x,y)\}\big\} \,.
\end{aligned} \tag{5.6}$$

But the inner 'supinf" is $(\mathrm{cl}_2K)(x,z)$, according to formula (2.4) again, and cl_2K is minimax equivalent to K because K is closed. The second half of (5.6) thus gives $\underline{L}(u,v)$. One can verify the other part of (5.4) in the same way. For the proof of the last statement in the theorem, we note that the upper conjugate $\bar{K}$ of any L minimax equivalent to $\bar{L}$ is the same as the upper conjugate of $\bar{L}$. Hence it is given by

$$\begin{aligned}
\bar{K}(x,y) &= \inf_u\sup_v\big\{(x,u)+(y,v)-\inf_z\sup_w\{(z,u)+(w,v)-K(z,w)\}\big\} \\
&= \inf_u\sup_z\big\{(x-z,u)+\sup_v\inf_w\{(y-w,v)+K(z,w)\}\big\} \,.
\end{aligned}$$

But the latter expression is $(\mathrm{cl}_1\mathrm{cl}_2K)(x,y)$ by (2.4) and its concave analog. Since K is closed, $\mathrm{cl}_1\mathrm{cl}_2K=\mathrm{cl}_1K$ is the upper member of the minimax equivalence class containing K. The parallel argument shows that the lower conjugate of L is in turn the lower member of this class.

REMARK. If K is not closed, the proof of Theorem 4 still shows that the upper conjugate $\bar{L}$ of K is a saddle-function, and that the upper conjugate of $\bar{L}$ is in turn $\mathrm{cl}_1 \mathrm{cl}_2 K$. The latter is always a closed minimax extension of K, as was demonstrated in the proof of Theorem 2. But $\bar{L}$ is not necessarily closed itself, when K is not closed, nor is it then always minimax equivalent to $\underline{L}$. As an example of this in the case $R^m = R = R^n$, one may take $K(x,y) = xy$ when $x > 0$ and $y \geqq 0$, $K(x,y) = +\infty$ when $x > 0$ and $y < 0$, $K(x,y) = -\infty$ when $x \leqq 0$. Here K is a simple proper saddle-function and $\underline{L}$ is closed, but $\bar{L}$ is not closed. It can be shown that, when K is simple, the closed minimax extensions of $\underline{L}$ and $\bar{L}$ are nevertheless all minimax equivalent, and their conjugates in turn give the class of closed minimax extensions of K. This is not true when K is not simple.

Any saddle-function L such that $\underline{L} \leqq L \leqq \bar{L}$ will be called simply a *conjugate* of K. Theorem 4 says that the conjugates of K are closed and minimax equivalent when K is closed. Furthermore, the conjugate relationship is symmetric and one-to-one among the minimax equivalence classes of closed saddle-functions on $R^m \times R^n$. In the improper case, the constant functions $-\infty$ and $+\infty$ are conjugate to one another. Therefore K is proper if and only if L is proper, when K and L are closed saddle-functions conjugate to one another.

If we combine Theorem 4 with the detailed description of closed saddle-functions in § 3, we get a wealth of facts about the nature of $\underline{L}$ and $\bar{L}$. In particular, part (c) of Theorem 1 yields the following important comparison of $\underline{L}$ and $\bar{L}$.

COROLLARY. *If K is a closed proper saddle-function on $R^m \times R^n$, there exist (unique) non-empty convex sets $C \subseteqq R^n$ and $D \subseteqq R^n$ such that the functions $\underline{L}$ and $\bar{L}$ in* (5.1) *satisfy the relationships in Table* 1 *(in place of $\underline{K}$ and $\bar{K}$).*

Observe from Corollary 1 to Theorem 1 that $\underline{L}$ and $\bar{L}$ depend only on the saddle-element $\{A, B, K\}$, where $A = \mathrm{dom}_1 K$ and $B = \mathrm{dom}_2 K$. The functions $\underline{L}$ and $\bar{L}$ thus are the same as the ones defined for closed saddle-elements in the introduction to this paper. Furthermore, the corollary just stated implies that the conjugates of $\{\mathrm{dom}_1 K, \mathrm{dom}_2 K, K\}$ (in the sense of the introduction) exist and are precisely the saddle-elements $\{\mathrm{dom}_1 L, \mathrm{dom}_2 L, L\}$ for the various conjugates L of K. Thus Theorem 4 furnishes the previously outlined facts about conjugate saddle-elements, via the correspondence between closed saddle-functions and their kernels that was set forth in Theorem 3 and Corollary 2 to Theorem 1.

The corollary above is essentially a minimax theorem. It describes and compares the lower and upper saddle-values $-\bar{L}(u,v)$ and $-\underline{L}(u,v)$ of the closed saddle-function $K(x,y)-(x,u)-(y,v)$ for the various possible choices of u and v. The sets C and D can of course be identified with $\mathrm{dom}_1 L$ and $\mathrm{dom}_2 L$ for any conjugate L of K.

These minimax results are interesting, at all events, in the qualitative sense. They show, for instance that the cases where the lower or upper saddle-value of a closed saddle-function is finite, but the two are not equal, are really quite exceptional. Such cases correspond to peculiar "discontinuities at corner points", like the behaviour of x^y on the unit square as pointed out in § 1.

To use these minimax results in a quantitative sense, however, one must have some way of determining the sets $\mathrm{dom}_1 L$ and $\mathrm{dom}_2 L$ without having to calculate a conjugate L of K directly. The following theorem characterizes all the closed half-spaces containing $\mathrm{dom}_1 L$ and $\mathrm{dom}_2 L$ in terms of simple properties of K. This at least provides a means of determining $\mathrm{cl}(\mathrm{dom}_1 L)$, $\mathrm{ri}(\mathrm{dom}_1 L)$, $\mathrm{cl}(\mathrm{dom}_2 L)$ and $\mathrm{ri}(\mathrm{dom}_2 L)$. Indeed, if C is a non-empty convex set, in R^m say, $\mathrm{cl}\, C$ is the intersection of all the closed half-spaces containing C. Thus $u_0 \in \mathrm{cl}\, C$ if and only if: $(x_0, u_0) \geqq \alpha_0$ whenever $x_0 \in R^m$ and $\alpha_0 \in R$ are such that $(x_0, u) \geqq \alpha_0$ for all $u \in C$. On the other hand, $u_0 \in \mathrm{ri}\, C$ if and only if:

$$(-x_0, u) \geqq -\alpha_0 \qquad \text{for all } u \in C$$

whenever

$$(x_0, u) \geqq \alpha_0 \geqq (x_0, u_0) \qquad \text{for all } u \in C\,.$$

This follows from the definition of relative interior, using the better known fact that a vector belongs to a given finite-dimensional open convex set if and only if it cannot be separated from the set by a non-zero hyperplane.

THEOREM 5. *Let K be a closed proper saddle-function on $R^m \times R^n$ and let L be any conjugate of K.*

(a) *$x_0 \in R^m$ and $\alpha_0 \in R$ have the property that $(x_0, u) \geqq \alpha_0$ for all $u \in \mathrm{dom}_1 L$ if and only if, for all $x \in \mathrm{ri}(\mathrm{dom}_1 K)$ and $y \in \mathrm{ri}(\mathrm{dom}_2 K)$,*

$$[K(x+\lambda x_0, y) - K(x,y)]/\lambda \geqq \alpha_0 \qquad \text{for all } \lambda > 0\,. \tag{5.7a}$$

(b) *$y_0 \in R^n$ and $\beta_0 \in R$ have the property that $(y_0, v) \leqq \beta_0$ for all $v \in \mathrm{dom}_2 L$ if and only if, for all $x \in \mathrm{ri}(\mathrm{dom}_1 K)$ and $y \in \mathrm{ri}(\mathrm{dom}_2 K)$,*

$$[K(x, y+\lambda y_0) - K(x,y)]/\lambda \leqq \beta_0 \qquad \text{for all } \lambda > 0\,. \tag{5.7b}$$

REMARK. When the second condition in (a) is satisfied, the half-line

$\{x+\lambda x_0 \mid \lambda \geqq 0\}$ is actually contained in $\mathrm{ri}(\mathrm{dom}_1 K)$ for every x $\mathrm{ri}(\mathrm{dom}_1 K)$. This is shown in the proof below. Thus Theorem 5 makes use only of the inner kernel of K. It can therefore be applied easily in the saddle-element context of § 1, in cases where it is more convenient to work in that notational scheme.

PROOF. Suppose (5.7a) holds for all $x \in \mathrm{ri}(\mathrm{dom}_1 K)$ and $y \in \mathrm{ri}(\mathrm{dom}_2 K)$. Fix any $\bar{x} \in \mathrm{ri}(\mathrm{dom}_1 K)$. Then $K(\bar{x}, y)$ is finite for $y \in \mathrm{dom}_2 K$. It follows from (5.7a) that $K(\bar{x}+\lambda x_0, y) > -\infty$ for all $y \in \mathrm{ri}(\mathrm{dom}_2 K)$ and $\lambda \geqq 0$, and hence that $\bar{x}+\lambda x_0 \in \mathrm{dom}_1' K$ by Lemma 1. But $\mathrm{dom}_1' K = \mathrm{dom}_1 K$ by Corollary 1 to Theorem 1. Also $\bar{x}$ is a relative interior point of $\mathrm{dom}_1 K$, so that the relative interior of any line segment connecting $\bar{x}$ with another point of $\mathrm{dom}_1 K$ will lie entirely in $\mathrm{ri}(\mathrm{dom}_1 K)$. Therefore

$$\bar{x}+\lambda x_0 \in \mathrm{ri}(\mathrm{dom}_1 K) \qquad \text{for all } \lambda \geqq 0 . \tag{5.8}$$

Now let $\underline{K}$ and $\bar{K}$ be the lower and upper members of the minimax equivalence class containing K. Then $\underline{K} = \mathrm{cl}_2 K \leqq K \leqq \bar{K}$. Hence $\underline{K}(x, \cdot)$ is a closed proper convex function with effective domain $\mathrm{dom}_2 K$ for each $x \in \mathrm{ri}(\mathrm{dom}_1 K)$, by part (c) of Theorem 1. The values of a closed convex function f at points of $\mathrm{cl}(\mathrm{dom} f)$ can always be expressed as limits of the values of f on $\mathrm{ri}(\mathrm{dom} f)$ using formula (2.6). We can conclude therefore from (5.7a) and (5.8) that

$$\underline{K}(\bar{x}+\lambda x_0, y) \geqq \underline{K}(\bar{x}, y)+\lambda \alpha_0 \qquad \text{for all } y \text{ and } \lambda \geqq 0 . \tag{5.9}$$

Now $\bar{K}$ is the upper conjugate of L by Theorem 4, so

$$\sup_v \{(\bar{x}+\lambda x_0, u)+(y, v)-L(u, v)\} \geqq \bar{K}(\bar{x}+\lambda x_0, y) \geqq \underline{K}(\bar{x}+\lambda x_0, y) \tag{5.10}$$

for all y and $\lambda \geqq 0$. Now choose any $z \in R^n$ and $\beta \in R$ such that

$$\underline{K}(\bar{x}, y) \geqq (y, z)+\beta \qquad \text{for all } y . \tag{5.11}$$

This is possible because $\underline{K}(\bar{x}, \cdot)$ is a closed proper convex function. Combining (5.9), (5.10) and (5.11), we get

$$(\bar{x}+\lambda x_0, u)+\sup_v \{(y, v)-L(u, v)\} \geqq (y, z)+\beta+\lambda \alpha_0$$

for all y and $\lambda \geqq 0$. Therefore

$$\begin{aligned} \lambda[(x_0, u)-\alpha_0]+(\bar{x}, u)-\beta &\geqq \sup_y \inf_v \{(y, z-v)+L(u, v)\} \\ &= (\mathrm{cl}_2 L)(u, z) = \underline{L}(u, z) \end{aligned}$$

by formula (2.4) for the closure of a convex function. This holds for all $\lambda \geqq 0$, so $\underline{L}(u, z) = -\infty$ when $(x_0, u)-\alpha < 0$. But for points $u \in \mathrm{dom}_1 \underline{L} = \mathrm{dom}_1 L$ we have $\underline{L}(u, z) > -\infty$ by definition. Thus $(x_0, u) \geqq \alpha_0$ for all

$u \in \mathrm{dom}_1 L$. Now assume the latter; We shall prove the "only if" part of (a). By Theorem 4, $\underline{K}$ is the lower conjugate of L. Calculating as in (4.2), we get

$$\begin{aligned} K(x+\lambda x_0, y) &\geqq \underline{K}(x+\lambda x_0, y) \\ &= \inf_v \sup\{(x+\lambda x_0, u)+(y,v)-L(u,v) \mid u \in \mathrm{dom}_1 L\} \\ &\geqq \inf_v \sup\{\lambda\alpha_0+(x,u)+(y,v)-L(u,v) \mid u \in \mathrm{dom}_1 L\} \\ &= \underline{K}(x,y)+\lambda\alpha_0 \end{aligned}$$

for arbitrary x, y and $\lambda \geqq 0$. When $x \in \mathrm{ri}(\mathrm{dom}_1 K)$ and $y \in \mathrm{ri}(\mathrm{dom}_2 K)$, we have $\underline{K}(x,y)=K(x,y)=\overline{K}(x,y)$ by part (c) of Theorem 1, so (5.7a) must hold. The proof of (b) is analogous.

6. Subgradient characterization of saddle-points.

Let K be any saddle-function on $R^m \times R^n$. For each $\bar{x} \in R^m$ and $\bar{y} \in R^n$, we denote by $\partial_1 K(\bar{x},\bar{y})$ the set of subgradients at $\bar{x}$ of the concave function $K(\cdot,\bar{y})$, as defined in § 2. We denote by $\partial_2 K(\bar{x},\bar{y})$ the set of subgradients at $\bar{y}$ of the convex function $K(\bar{x},\cdot)$. The product of the closed convex sets $\partial_1 K(\bar{x},\bar{y})$ and $\partial_2 K(\bar{x},\bar{y})$ will be denoted by $\partial K(\bar{x},\bar{y})$. According to the remarks in § 2, $\partial K(\bar{x},\bar{y})$ will be the singleton consisting of the ordinary gradient of K, if K happens to be finite and differentiable at $\langle x,y \rangle$. The following is a more general criterion for the existence of subgradients.

LEMMA 5. *Let K be a closed proper saddle-function. Then*

$$\partial_1 K(\bar{x},\bar{y}) \neq \emptyset \qquad \textit{for all } \bar{y} \textit{ whenever } \bar{x} \in \mathrm{ri}(\mathrm{dom}_1 K)\,,$$

and

$$\partial_2 K(\bar{x},\bar{y}) \neq \emptyset \qquad \textit{for all } \bar{x} \textit{ whenever } \bar{y} \in \mathrm{ri}(\mathrm{dom}_2 K)\,.$$

In particular $\partial K(\bar{x},\bar{y}) \neq \emptyset$ if $\langle \bar{x},\bar{y} \rangle$ is a relative interior point of $(\mathrm{dom}_1 K) \times (\mathrm{dom}_2 K)$.

PROOF. If $\bar{x} \in \mathrm{dom}_1 K$, then $K(\bar{x},\cdot)$ is a proper convex function, and the relative interior of its effective domain is $\mathrm{ri}(\mathrm{dom}_2 K)$. This follows from the relations in Table 1, via Theorem 1c, because K is closed and proper. Hence $\partial_2 K(\bar{x},\bar{y}) \neq \emptyset$ in this case for all $\bar{y} \in \mathrm{ri}(\mathrm{dom}_2 K)$ by (2.10). If $\bar{x} \notin \mathrm{dom}_1 K$, for similar reasons $K(\bar{x},\cdot)$ is an improper convex function having the value $-\infty$ through $\mathrm{ri}(\mathrm{dom}_2 K)$. Then $\partial_2 K(\bar{x},\bar{y}) = R^n$ trivially for all $\bar{y} \in \mathrm{ri}(\mathrm{dom}_2 K)$. The fact about $\partial_1 K(\bar{x},\bar{y})$ has a parallel proof, and the final assertion is just a combination of the two.

The theorem below relates the subgradients of a saddle-function K to those of a conjugate L. When the criterion in Lemma 5 is applied to L, the theorem yields sufficient conditions for the existence of saddle-points in minimax problems involving K. We shall put this fact to use in the next section.

THEOREM 6. *Let K and L be closed saddle-functions conjugate to one another. Then the following four conditions on a set of vectors $\bar{x}$, $\bar{y}$, $\bar{u}$ and $\bar{v}$ are equivalent*:

(a) $\langle \bar{u}, \bar{v} \rangle \in \partial K(\bar{x}, \bar{y})$,

(b) $\langle \bar{x}, \bar{y} \rangle \in \partial L(\bar{u}, \bar{v})$,

(c) $\langle \bar{x}, \bar{y} \rangle$ *is a saddle-point of* $K(x,y) - (x, \bar{u}) - (\bar{y}, v)$,

(d) $\langle \bar{u}, \bar{v} \rangle$ *is a saddle-point of* $L(u,v) - (\bar{x}, u) - (\bar{y}, v)$.

PROOF. Assume that (a) holds. By the definition of the subgradients, we have

$$
(6.1) \qquad \begin{aligned} K(x,\bar{y}) &\leqq K(\bar{x},\bar{y}) + (x - \bar{x}, \bar{u}) \qquad \text{for all } x\,, \\ K(\bar{x},y) &\geqq K(\bar{x},\bar{y}) + (y - \bar{y}, v) \qquad \text{for all } y\,. \end{aligned}
$$

Therefore, for $K_1(x,y) = K(x,y) - (x,\bar{u}) - (y,\bar{v})$,

$$
K_1(x,\bar{y}) \leqq K_1(\bar{x},\bar{y}) \leqq K_1(\bar{x},y)
$$

for all x and y. This is the same as (c). The argument can be reversed, so (a) is equivalent to (c). Hence (b) and (d) are equivalent, too. We shall now show that (a) implies (d). This will prove the theorem, because of the symmetry of the conjugate relationship. By definition, $L \leqq \bar{L}$, where $\bar{L}$ is the upper conjugate of K. Hence, for all u,

$$
(6.2) \qquad \begin{aligned} L(u,\bar{v}) \leqq \bar{L}(u,\bar{v}) &\leqq \sup_y \{(\bar{x},u) + (y,\bar{v}) - K(\bar{x},y)\} \\ &\leqq (\bar{x},u) + (\bar{y},\bar{v}) - K(\bar{x},\bar{y}) \end{aligned}
$$

by the second half of (6.1). Similarly

$$
(6.3) \qquad L(\bar{u},v) \geqq (x,\bar{u}) + (\bar{y},v) - K(\bar{x},\bar{y})
$$

for all v. Combining (6.2) and (6.3) we get

$$
L(u,\bar{v}) - (\bar{x},u) - (\bar{y},\bar{v}) \leqq L(\bar{u},v) - (\bar{x},\bar{u}) - (\bar{y},v)
$$

which is just (d).

COROLLARY. *If K and K' are closed saddle-functions in the same minimax equivalence class, then $\partial K(\bar{x},\bar{y}) = \partial K'(\bar{x},\bar{y})$ for all $\bar{x}$ and $\bar{y}$. Furthermore, K and K' have the same value at all points where these subgradient sets are non-empty.*

PROOF. Minimax equivalent saddle-function have the same saddle-points and saddle-values.

These results say that, for closed saddle-functions, the setvalued subgradient mappings

$$\langle \bar{x}, \bar{y} \rangle \to \partial K(\bar{x}, \bar{y})$$

depend only on minimax equivalence classes. The mapping for a given class is the "inverse" of the mapping for the class conjugate to it.

THEOREM 7. *Let K be a closed saddle-function on $R^m \times R^n$, and fix any $u \in R^m$ and $v \in R^n$. Then the infimum in (5.1b) is attained at $\bar{x}$ if and only if $\bar{x} \in \partial_1 \bar{L}(u,v)$. The supremum in (5.1a) is attained at $\bar{y}$ if and only if $\bar{y} \in \partial_2 \underline{L}(u,v)$.*

PROOF. Let $\bar{K}$ be the upper member of the minimax equivalence class of K. Then

$$\sup_y \{(x,y)+(y,v)-K(x,y)\} = \sup_y \{(x,u)+(y,v)-\bar{K}(x,y)\} \tag{6.4}$$

for all x by the definition of minimax equivalence. Denote the common value in (6.4) by $f(x)$. Since $\bar{K} = \mathrm{cl}_1 K$, $\bar{K}(\cdot, y)$ is a closed concave function for each y. The right side of (6.4) thus expresses f as a supremum of closed convex functions. Therefore f is itself a closed convex function. We must show that f attains its minimum at $\bar{x}$ if and only if $\bar{x} \in \partial \bar{L}(u,v)$. The latter means that

$$\bar{L}(w,v) \leqq \bar{L}(u,v) + (\bar{x}, w-u) \qquad \text{for all } w\,.$$

This is equivalent to

$$\bar{L}(u,v) = \sup_w \{\bar{L}(w,v) - (\bar{x}, w-u)\}\,. \tag{6.5}$$

The left side of (6.5) is $\inf_x f(x)$. The right side is

$$\begin{aligned} &\sup_w \{(x, u-w) + \inf_x \sup_y \{(x,w)+(y,v)-K(x,y)\}\} \\ &\quad = \sup_w \inf_x \{(\bar{x}-x, u-w) + f(x)\} \\ &\quad = (\mathrm{cl} f)(\bar{x}) = f(\bar{x})\,. \end{aligned}$$

The other half of the theorem has an analogous proof.

7. Minimax theorem.

We shall now give a formal statement of the special new minimax results referred to in § 1, and show that they are an elementary consequence of the general theory of conjugate saddle-functions. Our theorem makes use of the following two conditions on a saddle-element $\{A, B, K\}$.

(I) No non-zero vector x_0 has the property that, for all $x \in \operatorname{ri} A$ and $y \in \operatorname{ri} B$, the half-line $\{x+\lambda x_0 \mid \lambda \geqq 0\}$ is contained in A and $K(x+\lambda x_0, y)$ is a non-zero decreasing function of $\lambda \geqq 0$.

(II) No non-zero vector y_0 has the property that, for all $x \in \operatorname{ri} A$ and all $y \in \operatorname{ri} B$, the half-line $\{y+\lambda y_0 \mid \lambda \geqq 0\}$ is contained in B and $K(x, y+\lambda y_0)$ is a non-increasing function of $\lambda \geqq 0$.

Condition (I) is trivially satisfied, of course, if A is bounded. Indeed, it is easy to see that (I) is satisfied if, for merely one $y \in \operatorname{ri} B$ and $\alpha \in R$, $\{x \mid K(x,y) > \alpha\}$ is non-empty and bounded. The latter is very similar to the type of condition used by Moreau [4].

When $\{A, B, K\}$ is completely closed, the result below is a special case of another minimax theorem developed by the author [5, Theorem 4]. The previous theorem was proved by an entirely different method, entailing an extension of Helly's theorem. It makes stronger assertions in cases where polyhedral convex sets are involved in a certain way. Its proof cannot be extended, however, to the present case of general closed saddle-element.

THEOREM 8. *Let* $\{A, B, K\}$ *be any closed saddle-element on* $R^m \times R^n$. *If condition* (I) *is satisfied, then*

$$\max_{x \in A} \inf_{y \in B} K(x,y) = \inf_{y \in B} \sup_{x \in A} K(x,y) < +\infty . \tag{7.1}$$

If condition (II) *is satisfied, then*

$$-\infty < \sup_{x \in A} \inf_{y \in B} K(x,y) = \min_{y \in B} \sup_{x \in A} K(x,y) . \tag{7.2}$$

If (I) *and* (II) *are both satisfied,* K *has a saddle-point on* $A \times B$.

PROOF. By Theorem 3b we can assume that K is a closed proper saddle-function on $R^m \times R^n$, with $A = \operatorname{dom}_1 K$ and $B = \operatorname{dom}_2 K$. Let $\underline{L}$ and $\bar{L}$ be the lower and upper conjugate of K. Then

$$-\bar{L}(0,0) = \sup_{x \in A} \inf_{y \in B} K(x,y) , \tag{7.3a}$$

$$-\underline{L}(0,0) = \inf_{y \in B} \sup_{x \in A} K(x,y) \tag{7.3b}$$

by Corollary 1 to Theorem 1. Let

$$C = \operatorname{dom}_1 \underline{L} = \operatorname{dom}_1 \bar{L},\ D = \operatorname{dom}_2 \underline{L} = \operatorname{dom}_2 \bar{L} .$$

If $0 \in \operatorname{ri} C$, then $\underline{L}(0,0) = \bar{L}(0,0) > -\infty$ by the corollary to Theorem 4. Similarly, $\underline{L}(0,0) = \bar{L}(0,0) < +\infty$ if $0 \in \operatorname{ri} D$. According to Theorem 7, the supremum in (7.3a) is attained at $\bar{x}$ if and only if $\bar{x} \in \partial_1 \bar{L}(0,0)$. (Since

$\inf_y K(x,y)$ is always $-\infty$ for $x \notin A = \mathrm{dom}_1 K$, its supremum is attained on A if it is attained at all.) But $\partial_1 \bar{L}(0,0)$ is non-empty by Lemma 5, when $0 \in \mathrm{ri}\, C$. Thus the supremum in (7.3a) is attained when $0 \in \mathrm{ri}\, C$. The infimum in (7.3b) is likewise attained when $0 \in \mathrm{ri}\, D$. We need only observe finally that conditions (I) and (II) imply by Theorem 5 that $0 \in \mathrm{ri}\, C$ and $0 \in \mathrm{ri}\, D$, respectively. (They are actually equivalent to the origins being interior points of C and D.) The last assertion of the theorem merely combines the first two. It also follows immediately and independently from Lemma 5 and Theorem 6 via the same observation about conditions (I) and (II).

Example. Suppose that A and B are the non-negative orthants of R^m and R^n respectively, and that K is differentiable on the interior of A and B, that is, for $x \gg 0$ and $y \gg 0$. Denote by $\nabla_1 K(x,y)$ and $\nabla_2 K(x,y)$ the R^m and R^n components of the gradient of K at $\langle x,y \rangle$. Then (I) and (II) are equivalent to:

(I′) No non-zero $x_0 \geqq 0$ has the property that

$$\big(x_0, \nabla_1 K(x,y)\big) \geqq 0 \quad \text{for all } x \gg 0 \text{ and } y \gg 0;$$

(II′) No non-zero $y_0 \geqq 0$ has the property that

$$\big(y_0, \nabla_2 K(x,y)\big) \leqq 0 \quad \text{for all } x \gg 0 \text{ and } y \gg 0 .$$

These are certainly satisfied if there exist vectors $x_1 \gg 0$ and $y_1 \gg 0$ such that $\nabla_1 K(x_1,y_1) \ll 0$, and there exist vectors $x_2 \gg 0$ and $y_2 \gg 0$ such that $\nabla_2 K(x_2,y_2) \gg 0$.

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