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R. T. Rockafellar
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A MONOTONE CONVEX ANALOG OF LINEAR ALGEBRA

R. T. ROCKAFELLAR

1. Introduction.

The "monotone processes" defined below are essentially non-linear analogs of non-negative linear transformations. Monotone processes have inverses and adjoints. They can be added and multiplied together, etc. A significant "eigenvalue" theory can be built around them. In general, they enable one to construct an extensive "monotone convex algebra" parallel to real linear algebra. The new theory has been set forth by the writer in detail in [8] for eventual publication elsewhere. The present paper is an expository summary of the main ideas and results, without proofs.

In § 2, the duality between points and linear functions is elaborated into one between "monotone sets" and "monotone gauges". This sets the stage for the introduction in § 3 of "monotone processes" and their adjoints. The latter are developed using a generalized notion of "bilinear function". Combinatorial operations are discussed in § 5, and the "eigenvalue" results are discussed in § 6.

A remarkable feature of the theory is the way it presents new convex programming duality theorems as non-linear analogs of the classical formula defining the adjoint of a linear transformation. This is explained in § 4. Some possible applications to mathematical economics are touched upon briefly in § 6.

2. Monotone Sets and Gauges.

Let P_n denote the non-negative orthant of R^n, i.e. the set of all $x=(\xi_1,\ldots,\xi_n)\in R^n$ such that $\xi_j \geqq 0$ for $j=1,\ldots,n$. We write $x\geqq z$ if $x-z\in P_n$, $x>z$ if $x\geqq z$ but $x\neq z$, and $x\gg z$ if $x-z$ is an interior point of P_n. Thus $x\gg 0$ means that every component of x is positive. It is also useful to set

$$(2.1)\qquad x^{\wedge} = \{z \geqq 0 \mid z \leqq x\} \quad\text{and}\quad x^{\vee} = \{z \geqq 0 \mid z \geqq x\}\,.$$

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We define a *monotone set of concave type* in P_n to be a non-empty closed subset C of P_n such that

$$x \in C \quad \text{implies} \quad x^\wedge \subseteq C\,. \tag{2.2a}$$

A *monotone set of convex type* in P_n, on the other hand, is a non-empty closed (unbounded) convex subset C of P_n such that

$$x \in C \quad \text{implies} \quad x^\vee \subseteq C\,. \tag{2.2b}$$

In particular, for any $x \in P_n$, $x^\wedge$ and $x^\vee$ are monotone sets of concave type and of convex type, respectively. In the discussion which follows these sets provide the bridge between statements involving vectors and statements involving general monotone sets.

The vector sum $C_1 + C_2$ of two monotone sets of the same type is another such set. Likewise, the scalar multiple λC is a monotone set of the same type as C if $\lambda > 0$. It is convenient accordingly to set $0 \cdot C = \{0\}$ when C is of concave type, but $0 \cdot C = P_n$ when C is of convex type. Addition and non-negative scalar multiplication of monotone sets satisfy

$$\begin{aligned} \lambda(C_1 + C_2) &= \lambda C_1 + \lambda C_2\,,\\ (\lambda_1 + \lambda_2)C &= \lambda_1 C + \lambda_2 C\,,\\ \lambda_1(\lambda_2 C) &= (\lambda_1 \lambda_2)C\,. \end{aligned} \tag{2.3}$$

It is natural to define $C_1 \geqq C_2$ to mean $C_1 \supseteq C_2$ when both monotone sets are of concave type, and to define it to mean $C_1 \subseteq C_2$ when both are of convex type. When C_1 is of concave type and C_2 is of convex type, we let $C_1 \geqq C_2$ mean that $C_1 \cap C_2 \neq \emptyset$ (see below).

A *monotone gauge of convex type* on P_n is a continuous non-negative real-valued function f such that

$$\begin{aligned} f(x_1) &\geqq f(x_2) && \text{for } x_1 \geqq x_2 \geqq 0\,,\\ f(\lambda x) &= \lambda f(x) && \text{for } \lambda \geqq 0 \text{ and } x \geqq 0\,,\\ f(x_1 + x_2) &\leqq f(x_1) + f(x_2) && \text{for } x_1 \geqq 0,\ x_2 \geqq 0\,. \end{aligned} \tag{2.4}$$

A *monotone gauge of concave type* is defined in the same way, except that the inequality in the last condition of (2.4) is reversed. The functions which are monotone gauges of both types simultaneously are precisely the non-negative *linear* functions. These correspond canonically to n-tuples of non-negative coefficients and hence could be identified with the elements of P_n itself. Rather than make this identification, we conceive of the coefficient n-tuples as belonging to $P_n{}^*$, a second copy of the non-negative orthant of R^n, and we write

$$\langle x, x^* \rangle = \xi_1 \xi_1{}^* + \ldots + \xi_n \xi_n{}^* \tag{2.5}$$

for $x = (\xi_1, \ldots, \xi_n) \in P_n$ and $x^* = (\xi_1{}^*, \ldots, \xi_n{}^*) \in P_n{}^*$.

The elements of P_n are in turn viewed as the coefficient n-tuples for the linear monotone gauges on $P_n{}^*$. An asterisk will generally be used to indicate that an object belongs to the dual orthant $P_n{}^*$ rather than to P_n.

Let C be a monotone set in P_n. We define the *monotone support function* $\langle C, \cdot\rangle$ of C on $P_n{}^*$ by

$$\langle C,x^*\rangle = \sup\{\langle x,x^*\rangle \mid x \in C\} \tag{2.6a}$$

if C is of concave type, and by

$$\langle C,x^*\rangle = \inf\{\langle x,x^*\rangle \mid x \in C\} \tag{2.6b}$$

if C is of convex type. The following analog of the classical theorem about convex sets and their support functions turns out to be valid: *the monotone support function $\langle C, \cdot\rangle$ of a monotone set C of concave type in P_n is a monotone gauge of convex type on $P_n{}^*$, and every such function on $P_n{}^*$ arises this way from a unique C.* Likewise, the monotone sets of convex type in P_n correspond one-to-one with the monotone gauges of concave type on $P_n{}^*$. The monotone gauges on P_n correspond dually to the monotone sets C^* in $P_n{}^*$.

With respect to addition and non-negative scalar multiplication of monotone sets, one has

$$\begin{aligned} \langle C_1+C_2, x^*\rangle &= \langle C_1, x^*\rangle+\langle C_2, x^*\rangle\,, \\ \langle \lambda C, x^*\rangle &= \lambda\langle C, x^*\rangle\,. \end{aligned} \tag{2.7}$$

The definition of $\geqq$ for monotone sets is equivalent in all cases to

$$C_1 \geqq C_2 \quad \text{if and only if} \quad \langle C_1, x^*\rangle \geqq \langle C_2, x^*\rangle \quad \text{for all } x^*. \tag{2.8}$$

Our suggestive notation for monotone support functions can be carried a step further. It can be proved that

$$\sup_{x\in C}\ \inf_{x^*\in D^*} \langle x, x^*\rangle = \inf_{x^*\in D^*}\ \sup_{x\in C} \langle x, x^*\rangle \tag{2.9}$$

for any monotone set C of concave type in P_n and any monotone set D^* of convex type in $P_n{}^*$. We shall denote by $\langle C, D^*\rangle$ the non-negative real number represented in (2.9). Thus by definition

$$\langle C, D^*\rangle = \sup_{x\in C} \langle x, D^*\rangle = \inf_{x^*\in D^*} \langle C, x^*\rangle\,. \tag{2.10}$$

This "pairing" satisfies

$$\begin{aligned} &\langle C_1, D_1{}^*\rangle \geqq \langle C_2, D_2{}^*\rangle \quad \text{if } C_1 \geqq C_2 \text{ and } D_1{}^* \geqq D_2{}^*, \\ &\langle \lambda C, D^*\rangle = \lambda\langle C, D^*\rangle = \langle C, \lambda D^*\rangle \quad \text{if } \lambda \geqq 0\,, \end{aligned} \tag{2.11}$$

$$\langle C_1+C_2, D^*\rangle \geqq \langle C_1, D^*\rangle+\langle C_2, D^*\rangle ,$$
$$\langle C, D_1^*+D_2^*\rangle \leqq \langle C, D_1^*\rangle+\langle C, D_2^*\rangle .$$

When the types of the sets are reversed, the "inf" and "sup" abo must naturally be reversed, too. The last two inequalities in (2.11) a then to be reversed.

A monotone set C of concave type in P_n will be called *non-singul* if it has a non-empty interior. The formulas

(2.12a) $C^* = \{x^* \geqq 0 \mid \langle C, x^*\rangle \leqq 1\}$ and $C = \{x \geqq 0 \mid \langle x, C^*\rangle \leqq$

define a one-to-one *polar* correspondence between the non-singul monotone sets of concave type in P_n and those in $P_n{}^*$. On the oth hand, a monotone set C of *convex* type in P_n will be called non-singula if it is not all of P_n. A polar correspondence between such sets in P and in $P_n{}^*$ is set up by

(2.12b) $C^* = \{x^* \geqq 0 \mid \langle C,x^*\rangle \geqq 1\}$ and $C = \{x \geqq 0 \mid \langle x,C^*\rangle \geqq 1\}$

If C and D are non-singular monotone sets of opposite types in P_n and if C^* and D^* are their polars in $P_n{}^*$, then

(2.13) $$\langle C,D^*\rangle\cdot\langle D,C^*\rangle = 1 .$$

3. Monotone Processes.

We define a *monotone process of concave type* from P_n to P_m to b a multivalued function T which associates with each $x \in P_n$ a *monoton set* $T(x)$ of concave type in P_n in such a way that

(a) $T(x_1) \geqq T(x_2)$ if $x_1 \geqq x_2$,
(b) $T(\lambda x) = \lambda T(x)$ for $\lambda \geqq 0$,
(c) $T(x_1+x_2) \geqq T(x_1)+T(x_2)$,
(d) If $y_i \in T(x_i)$ for $i=1,2,\dots,x_i \to x$ and $y_i \to y$, then $y \in T(x)$.

A *monotone process of convex type* is defined in the same way, except tha $T(x)$ is required to be of convex type, and the inequality in conditio (c) is reversed. Examples will be given in the next section.

The *inverse* of a monotone process T from P_n to P_m is the multi valued function T^{-1} from P_m back to P_n defined by

(3.1) $$T^{-1}(y) = \{x \geqq 0 \mid y \in T(x)\} \quad \text{for each } y \geqq 0 .$$

Investigation of T^{-1} leads one to call a T of concave type *non-singula* if $T(x)$ has a non-empty interior when $x \gg 0$; on the other hand, on calls a monotone process T of convex type *non-singular* if $0 \notin T(x$

when $x \neq 0$. One has the following theorem: *the inverse of a non-singular monotone process is a non-singular monotone process of the opposite type.*

In linear algebra, the correspondence between linear transformations and bilinear functions is crucial. Here we may consider, instead of bilinear functions, the *monotone bi-gauges of concave-convex type* on $P_n \times P_m^*$, i.e. the functions K on $P_n \times P_m^*$, such that $K(\cdot, y^*)$ is a monotone gauge of concave type on P_n for each fixed $y^* \in P_m^*$, and $K(x, \cdot)$ is a monotone gauge of convex type on P_n^* for each fixed $x \in P_n$. On the other hand, given a monotone process T of concave type from P_n to P_m, we can define a function $\langle T(\cdot), \cdot \rangle$ on $P_n \times P_m^*$ by (2.8), namely

$$\langle T(x), y^* \rangle = \sup \{ \langle y, y^* \rangle \mid y \in T(x) \} . \tag{3.2}$$

We call $\langle T(\cdot), \cdot \rangle$ the *Kuhn–Tucker function* of T, because of its significance in the programming theory to be explained in the next section. The following theorem may be proved: *The Kuhn–Tucker function $\langle T(\cdot), \cdot \rangle$ of a monotone process T of concave type from P_n to P_m is a monotone bi-gauge of concave-convex type on $P_n \times P_m^*$, and every such function on $P_n \times P_m^*$ arises this way from a unique T.*

In like manner, the monotone processes of *convex* type from P_n to P_m correspond one-to-one with the monotone bi-gauges of *convex-concave* type on $P_n \times P_m^*$ (with "sup" replaced by "inf" of course in (3.2)).

The facts just stated enable us to define, in the classical way, the *adjoint* T^* of a monotone process T from P_n to P_m. We take T^* to be the unique monotone process from P_m^* to P_n^* such that

$$\langle T(x), y^* \rangle = \langle x, T^*(y^*) \rangle \quad \text{for all } x \in P_n \text{ and } y^* \in P_m^* . \tag{3.3}$$

Note that T^* is of the opposite type from T, and $(T^*)^* = T$.

Adjoints can usually be calculated directly, without explicit intervention of the Kuhn–Tucker function. For instance, if T is of concave type, $T^*(y^*)$ will consist of the vectors $x^* \geqq 0$ such that

$$\langle x, x^* \rangle \geqq \langle y, y^* \rangle \quad \text{whenever } x \geqq 0 \text{ and } y \in T(x) . \tag{3.4}$$

It turns out that a monotone process T from P_n to P_m is non-singular if and only if its adjoint T^* is non-singular, in which case

$$T^{*-1} = T^{-1*} . \tag{3.5}$$

The latter monotone process, which goes from P_n^* to P_m^* is of the *same* type as T. We call it the *polar* of T.

The Kuhn–Tucker function of T^{-1} is related to that of T by a minimax formula:

$$(3.6)\qquad \langle T^{-1}(y),x^*\rangle = \inf_{x\geq 0}\sup_{y^*\geq 0}\{\langle x,x^*\rangle+\langle y,y^*\rangle-\langle T(x),y^*\rangle\}$$
$$= \sup_{y^*\geq 0}\inf_{x\geq 0}\{\langle x,x^*\rangle+\langle y,y^*\rangle-\langle T(x),y^*\rangle\}$$

when T is of concave type. (If T is of convex type, the roles of the "inf" and "sup" must be reversed.)

Given a monotone set C in P_n and a monotone process T from P_n to P_m of the same type as C, we define

$$(3.7)\qquad T(C) = \operatorname{cl} \bigcup_{x\in C} T(x)\,,$$

where "cl" stands for closure. Then $T(C)$ is a monotone set in P_m of the same type as C. Actually, the closure operation in (3.7) is superfluous if T is non-singular or of concave type. If C and T are non-singular, so is $T(C)$, and its polar is the image of the polar of C under the polar of T; in symbols

$$(3.8)\qquad T(C)^* = T^{*-1}(C^*)\,.$$

If D^* is a monotone set in $P_m{}^*$ of opposite type from C, one has

$$(3.9)\qquad \langle T(C),D^*\rangle = \langle C,T^*(D^*)\rangle\,.$$

A higher version of (3.6) holds similarly: if C^* and D are non-singular monotone sets of opposite type in $P_n{}^*$ and P_m, respectively, and if T is a non-singular monotone process from P_n to P_m of opposite type from D, one has

$$(3.10)\qquad \langle T^{-1}(D),C^*\rangle = \operatorname*{minimax}_{x\geq 0,\, y^*\geq 0}\{\langle x,C^*\rangle+\langle D,y^*\rangle-\langle T(x),y^*\rangle\}$$

where one minimizes in the convex argument and maximizes in the concave argument.

4. Examples and Applications to Convex Programming.

Each non-negative $m\times n$ matrix A corresponds to a certain monotone process T of concave type from P_n to P_m, namely

$$(4.1)\qquad T(x) = \{y \geqq 0 \mid y \leqq Ax\} = (Ax)^\wedge \quad \text{for each } x \geqq 0\,.$$

The Kuhn–Tucker function of T is obviously the non-negative restriction of the bilinear function defined by A, so

$$(4.2)\qquad \langle x,T^*(y^*)\rangle = \langle T(x),y^*\rangle = \langle Ax,y^*\rangle = \langle x,A^*y^*\rangle\,,$$

where A^* is the transpose of A. Therefore the adjoint T^* of T is the monotone process of convex type from $P_m{}^*$ to $P_n{}^*$ corresponding to A^*,

$$T^*(y^*) = \{x^* \geqq 0 \mid x^* \geqq A^*y^*\} = (A^*y^*)^\vee \quad \text{for each } y^* \geqq 0 . \tag{4.3}$$

Assume now that no row of A consists entirely of zeros. Then T and T^* are non-singular, and the inverses given by

$$\begin{aligned} T^{-1}(y) &= \{x \geqq 0 \mid Ax \geqq y\} \quad \text{for each } y \geqq 0 , \\ T^{*-1}(x^*) &= \{y^* \geqq 0 \mid A^*y^* \leqq x^*\} \quad \text{for each } x^* \geqq 0 , \end{aligned} \tag{4.4}$$

are monotone processes of convex and concave types, respectively. For a fixed $y \in P_m$ and a fixed x^* in P_n^*, each of the sets in (4.4) consists of the solutions to a certain finite system of linear inequalities. The formula

$$\langle T^{-1}(y), x^*\rangle = \langle y, T^{-1*}(x^*)\rangle = \langle y, T^{*-1}(x^*)\rangle \tag{4.5}$$

says that for fixed y and x^*

$$\inf\{\langle x,x^*\rangle \mid x \geqq 0,\ Ax \geqq y\} = \sup\{\langle y,y^*\rangle \mid y^* \geqq 0,\ A^*y^* \leqq x^*\} . \tag{4.6}$$

This is the "monotone" case of the famous linear programming duality theorem of Gale, Kuhn and Tucker. By (3.6), the common extremum in (4.6) is also the minimax of

$$\langle x,x^*\rangle + \langle y,y^*\rangle - \langle Ax,y^*\rangle \quad \text{for } x \geqq 0 \text{ and } y^* \geqq 0 . \tag{4.7}$$

This characterization is also well known in linear programming theory.

Important classes of *non-linear* programming problems can be viewed in this new way too. Let $f_1, \ldots, f_m$ be monotone gauges of concave type on P_n, and let g be a monotone gauge of convex type on P_n. For each non-negative choice of the constants $\eta_1, \ldots, \eta_m$, one may consider the problem

$$\begin{gathered} \text{minimize } g(\xi_1, \ldots, \xi_n) \text{ subject to} \\ \xi_j \geqq 0 \text{ for } j = 1, \ldots, m \text{ and } f_i(\xi_1, \ldots, \xi_n) \geqq \eta_i \text{ for } i = 1, \ldots, m . \end{gathered} \tag{4.8}$$

If we define T by

$$T(x) = \{y = (\eta_1, \ldots, \eta_m) \mid 0 \leqq \eta_i \leqq f_i(x) \quad \text{for } i = 1, \ldots, m\} , \tag{4.9}$$

T is a monotone process of concave type from P_n to P_m whose Kuhn-Tucker function is

$$\langle T(x), y^*\rangle = \sum_{i=1}^{m} \eta_i^* f_i(x) \quad \text{for } y^* = (\eta_1^*, \ldots, \eta_m^*) . \tag{4.10}$$

Assuming that no f_i is identically zero, so that T is non-singular, we can restate (4.8) as

$$\text{minimize } g \text{ on } T^{-1}(y) . \tag{4.8$'$}$$

Now let C^* be the unique monotone set of convex type in $P_n{}^*$ such that $g(x) = \langle x, C^*\rangle$ for all x in P_n. We have

$$(4.11)\qquad \begin{aligned}\inf\{g(x) \mid x \in T^{-1}(y)\} &= \langle T^{-1}(y), C^*\rangle = \langle y, T^{-1*}(C^*)\rangle \\ &= \langle y, T^{*-1}(C^*)\rangle = \sup\{\langle y, y^*\rangle \mid T^*(y^*) \cap C^* \neq \emptyset\}.\end{aligned}$$

This formula evidently can be interpreted as a *convex programming duality theorem*. Observe from (3.10) that the common extremum in (4.11) can also be expressed as the minimax in $x \geqq 0$ and $y^* \geqq 0$ of

$$(4.12)\qquad \langle x, C^*\rangle + \langle y, y^*\rangle - \langle T(x), y^*\rangle = g(x) + \textstyle\sum_i \eta_i{}^*(\eta_i - f_i(x)).$$

The Kuhn–Tucker theory of Lagrange multipliers for problem (4.8) may be derived this way. In order to get more insight into the nature of the dual problem here, we must calculate T^*. For $i = 1, \ldots, m$, let $C_i{}^*$ be the unique monotone set of convex type in $P_n{}^*$ such that $\langle x, C_i{}^*\rangle = f_i(x)$ for all x in P_n.

$$(4.13)\quad \langle x, T^*(y^*)\rangle = \langle T(x), y^*\rangle = \textstyle\sum_i \eta_i{}^*\langle x, C_i{}^*\rangle = \langle x, \sum_i \eta_i{}^* C_i{}^*\rangle,$$

and therefore

$$(4.14)\quad T^*(y^*) = \textstyle\sum_i \eta_i{}^* C_i{}^* \quad \text{for each } y^* = (\eta_1{}^*, \ldots, \eta_m{}^*) \geqq 0.$$

Using our definition of $\leqq$ when the "smaller" set is of convex type and the "larger" set is of concave type, we can now express the extremum problem at the end of (4.11) by

$$(4.15)\qquad \begin{gathered}\text{maximize } \eta_1\eta_1{}^* + \ldots + \eta_m\eta_m{}^* \text{ subject to} \\ \eta_1{}^* \geqq 0, \ldots, \eta_m{}^* \geqq 0, \quad \eta_1{}^*C_1{}^* + \ldots + \eta_m{}^*C_m{}^* \leqq C^*.\end{gathered}$$

This is the problem dual to (4.8). Problems of this form are also important in convex programming (see [1, Chap. 22]), but the duality displayed here seems to be new.

In general, suppose T is a non-singular monotone process of concave type from P_n to P_m. Let C^* be a monotone set of concave type in $P_n{}^*$, and let D be a monotone set of convex type in P_m. We may then consider the dual pair of monotone non-linear programs

$$(4.16\text{a})\qquad \text{minimize } \langle x, C^*\rangle \text{ subject to } x \geqq 0,\, T(x) \geqq D,$$

$$(4.16\text{b})\qquad \text{maximize } \langle D, y^*\rangle \text{ subject to } y^* \geqq 0,\, T^*(y^*) \leqq C^*.$$

The infimum in the first problem is $\langle T^{-1}(D), C^*\rangle$, while the supremum in the second problem is $\langle D, T^{*-1}(C^*)\rangle$. The two extrema are therefore equal by (3.5) and (3.9), and they also coincide with the minimax in (3.10). It can be shown that a pair of vectors $\bar{x} \geqq 0$ and $\bar{y}^* \geqq 0$ is a saddlepoint in (3.10) if and only if $\bar{x}$ is a solution to (4.16a) and $\bar{y}^*$ is a solution to (4.16b). Novel "polarity" and "reciprocity" theorems, having no counterpart in ordinary convex programming theory, can also be proved in this context. These relate the solutions and extrema in problems (4.16a) and (4.16b) to those in the corresponding "polar problems", i.e. where the elements T, T^*, C^*, D, are replaced by their polars.

A noteworthy case is the following. Let $C_1^*, \ldots, C_r^*$ be monotone sets of convex type in P_n^*, and let $D_1, \ldots, D_r$ be monotone sets of concave types in P_m. Define T by

$$T(x) = \langle x, C_1^* \rangle D_1 + \ldots + \langle x, C_r^* \rangle D_r \, . \tag{4.17}$$

Then T is a monotone process of concave type from P_n to P_m. The Kuhn–Tucker function of T is given by

$$\langle T(x), y^* \rangle = \sum_{k=1}^{r} \langle x, C_k^* \rangle \cdot \langle D_k, y^* \rangle \, , \tag{4.18}$$

and hence the monotone process of convex type adjoint to T is given by

$$T^*(y^*) = \langle D_1, y^* \rangle C_1^* + \ldots + \langle D_r, y^* \rangle C_r^* \, . \tag{4.19}$$

Finally, we would like to point out that the monotone processes from P_n to P_1 correspond, as expected, to the monotone gauges on P_n. The monotone processes from P_n to P_1, on the other hand, are of the form

$$\xi \to \xi C \quad \text{for } \xi \geqq 0 \, ,$$

where C is a monotone set in P_n. Such a monotone process is non-singular if and only if the set C is non-singular, in which case the polar process form P_1^* to P_n^* is given by

$$\xi^* \to \xi^* C^* \quad \text{for } \xi^* \geqq 0 \, ,$$

where C^* is the polar of C.

5. Combinatorial Operations.

Let T_1 and T_2 be monotone processes of the same type from P_n to P_m. We may then define $T_1 + T_2$ by

$$(T_1 + T_2)(x) = T_1(x) + T_2(x) \, . \tag{5.1}$$

It turns out that $T_1 + T_2$ is another monotone process from P_n to P_m, and

$$(T_1 + T_2)^* = T_1^* + T_2^* \, . \tag{5.2}$$

Non-negative scalar multiples of monotone processes can also be defined in the obvious way by

$$(\lambda T)(x) = \lambda T(x) . \tag{5.3}$$

Less obvious is the fact that the operation "$\square$" defined by

$$(T_1 \square T_2)(x) = \bigcup \{T_1(x-z) \cap T_2(z) \mid 0 \leqq z \leqq x\} \tag{5.4}$$

yields another monotone process of the same type as T_1 and T_2. Like addition, this binary operation is commutative, associative, and satisfies

$$(T_1 \square T_2)^* = T_1^* \square T_2^* . \tag{5.5}$$

We call it *inverse addition*, because

$$(T_1 \square T_2)^{-1} = T_1^{-1} + T_2^{-1} \quad \text{and} \quad (T_1 + T_2)^{-1} = T_1^{-1} \square T_2^{-1} \tag{5.6}$$

when T_1 and T_2 are non-singular.

The partial ordering for monotone sets may be turned into one for monotone processes by defining

$$T_1 \geqq T_2 \quad \text{if } T_1(x) \geqq T_2(x) \text{ for all } x . \tag{5.7}$$

Under this ordering, the set of all monotone processes of a given type from P_n to P_m is actually a *conditionally complete lattice*. For monotone processes of concave type, the (commutative associative binary) lattice operations are given by

$$\begin{aligned} (T_1 \wedge T_2)(x) &= T_1(x) \cap T_2(x) , \\ (T_1 \vee T_2)(x) &= \bigcup \{T_1(x-z) + T_2(z) \mid 0 \leqq z \leqq x\} . \end{aligned} \tag{5.8}$$

These formulas must be reversed for monotone processes of convex type.

The adjoint operation is order-preserving, and hence

$$(T_1 \wedge T_2)^* = T_1^* \wedge T_2^* \quad \text{and} \quad (T_1 \vee T_2)^* = T_1^* \vee T_2^* . \tag{5.9}$$

The inverse operation is *ordering-inverting*, so that

$$(T_1 \wedge T_2)^{-1} = T_1^{-1} \vee T_2^{-1} \quad \text{and} \quad (T_1 \vee T_2)^{-1} = T_1^{-1} \wedge T_2^{-1} \tag{5.10}$$

in the non-singular case. The operations of addition, inverse addition and non-negative scalar multiplication are monotone with respect to the partial ordering.

Perhaps the most interesting operation is *binary multiplication*. The product ST is defined, for a monotone process T from P_n to P_m and a monotone process S from P_m to P_r of the same type as T, by

$$(ST)(x) = S(T(x)) . \tag{5.11}$$

(See (3.7) and the remarks following it.) Since

$$\langle S(T(x)), z^*\rangle = \langle T(x), S^*(z^*)\rangle = \langle x, T^*(S^*(z^*))\rangle \,, \tag{5.12}$$

we have

$$(ST)^* = T^*S^* \,. \tag{5.13}$$

If S and T are non-singular, so is ST and

$$(ST)^{-1} = T^{-1}S^{-1} \,. \tag{5.14}$$

As in linear algebra, binary multiplication is associative (but not commutative). It is also monotone, i.e. $S_1T_1 \geqq S_2T_2$ when $S_1 \geqq S_2$ and $T_1 \geqq T_2$.

Although binary multiplication is not distributive across addition, it does satisfy certain *distributive inequalities*. For example, with monotone processes of concave type one always has

$$S(T_1+T_2) \geqq ST_1+ST_2 \quad \text{and} \quad (S_1+S_2)T \leqq S_1T+S_2T \,. \tag{5.15}$$

With convex types, these inequalities have to be reversed. Similar distributive inequalities hold between binary multiplication and inverse addition.

Particular attention will be devoted in the next section to the set of all monotone processes of a given type from P_n to itself. This is a conditionally complete lattice supplied with three further monotone associative binary operations (addition, inverse addition and multiplication) and a non-negative scalar multiplication. The sequence of powers

$$T, T^2, \ldots, T^k, \ldots$$

can then be defined and studied from the point of view of semigroup theory.

One may derive formulas which show how the Kuhn–Tucker functions behave under the various combinatorial operations. These formulas ought to be useful, for instance, in convex programming applications.

6. Sub-eigenvalues and Eigensets.

Throughout this section, let T denote a non-singular monotone process from P_n to itself. A positive real number λ is called a *sub-eigenvalue* of T if

$$\lambda x \in T(x) \quad \text{for some } x > 0 \,. \tag{6.1}$$

This condition is quite weak; in particular, every $x \gg 0$ satisfies (6.1) for some $\lambda > 0$ by non-singularity. Interest therefore centers on chaterizing the set of sub-eigenvalues of T as a whole, and on relating it to the corresponding set for T^*. The sub-eigenvalues of T^{-1} are, of course, just the reciprocals of those of T.

If T is of concave type, we define its *upper growth rate* $\bar{\lambda}$ and its *lower growth rate* $\underline{\lambda}$ by

$$\begin{aligned} \bar{\lambda} &= \sup\{\lambda > 0 \mid \lambda x \in T(x) \text{ for some } x > 0\}, \\ \underline{\lambda} &= \sup\{\lambda > 0 \mid \lambda x \in T(x) \text{ for some } x \gg 0\}. \end{aligned} \tag{6.2}$$

If T is of convex type, we instead set

$$\begin{aligned} \bar{\lambda} &= \inf\{\lambda > 0 \mid \lambda x \in T(x) \text{ for some } x \gg 0\}, \\ \underline{\lambda} &= \inf\{\lambda > 0 \mid \lambda x \in T(x) \text{ for some } x > 0\}. \end{aligned} \tag{6.3}$$

In both cases, it turns out that

$$0 < \underline{\lambda} \leqq \bar{\lambda} < \infty. \tag{6.4}$$

We say T is *evenly growing* if $\bar{\lambda} = \underline{\lambda}$. The terminology is suggested by certain economic applications and limit theorems which will be discussed below.

The main results about sub-eigenvalues are the following. Suppose for definiteness that T is of concave type, so that T^* is of convex type. The upper growth rates of T and T^* then coincide and are given by

$$\bar{\lambda} = \inf_{x^* \gg 0} \sup_{x \gg 0} \frac{\langle T(x), x^* \rangle}{\langle x, x^* \rangle}. \tag{6.5}$$

The lower growth rates of T and T^* likewise coincide and are given by

$$\underline{\lambda} = \sup_{x \gg 0} \inf_{x^* \gg 0} \frac{\langle T(x), x^* \rangle}{\langle x, x^* \rangle}. \tag{6.6}$$

Furthermore, λ is a sub-eigenvalue of T if and only if $0 < \lambda \leqq \bar{\lambda}$, while λ is a sub-eigenvalue of T^* if and only if $\underline{\lambda} \leqq \lambda < \infty$. In particular, *the set of sub-eigenvalues* λ *common to* T *and* T^* *is the non-empty closed bounded positive interval* $[\underline{\lambda}, \bar{\lambda}]$.

There is an important corollary: T is evenly growing if and only if T^* is evenly growing. In that case there is a *unique* sub-eigenvalue λ common to T and T^*, and it is given by

$$\lambda = \operatorname*{minimax}_{x^* \gg 0,\, x \gg 0} \frac{\langle T(x), x^* \rangle}{\langle x, x^* \rangle}. \tag{6.7}$$

A notable class of monotone processes which are always evenly growing are the *positive processes*, i.e. the ones such that $\langle T(x), x^* \rangle$ is positive except when $x = 0$ or $x^* = 0$.

The results just described are analogs of the Perron–Frobenius theory of positive matrices. Indeed, for any $n \times n$ matrix A of positive real numbers, the corresponding monotone process T of concave type from P_n to itself defined in (4.1) is positive, and hence is evenly growing. The special sub-eigenvalue λ in (6.7) is easily seen to be the sole positive eigenvalue of A.

The results about sub-eigenvalues can also be interpreted in light of economic growth rate theories, such as the one developed by Gale in [3]. Suppose that the vectors x in P_n are thought of as representing the possible goods states of an economy. For each x let $T(x)$ consist of the various states into which x could be transformed in one time unit. Under reasonable economic assumptions, T is a non-singular monotone process of concave type. Now $\lambda > 0$ is a sub-eigenvalue of T if and only if some goods state $x > 0$ can be transformed into λ times itself. One is naturally interested in the largest such λ, which is $\bar{\lambda}$ or $\underline{\lambda}$ if one insists on having $x \gg 0$. An element x^* of $P_n{}^*$ may be interpreted as a price or valuation vector. Then $\langle x, x^* \rangle$ gives the total value present in the economy when the goods state is x and the "market state" is x^*. Similarly

$$\langle T(x), x^* \rangle = \sup\{\langle y, x^* \rangle \mid y \in T(x)\} \tag{6.8}$$

gives the highest possible value attainable after one time unit, if the present goods state is x and the future market state is x^*. Since (6.8) also equals $\langle x, T^*(x^*) \rangle$, T^* can be thought of as a valuation mechanism which converts future values into present values ("shadow prices"). The inverse T^{*-1} (the polar of T) thus gives the set of (theoretical) market states into which a market state x^* may be transformed by the economy in one time unit. The condition $\lambda x^* \in T^*(x^*)$, for $\lambda > 0$ and $x^* > 0$, is equivalent to $\lambda^{-1} x^* \in T^{*-1}(x^*)$, which says that a certain market state x^* can be transformed into λ^{-1} times itself. One is interested in the largest such λ^{-1}, or equivalently in the smallest sub-eigenvalue λ of T^*, which is $\underline{\lambda}$. In case T is evenly growing, we may conclude that the largest possible growth rate for market states x^* is the reciprocal of the largest possible growth rate λ for goods states x. (As goods become more abundant their values per unit go down, or inversely.) Formula (6.7) points to a game-theoretic equilibrium involving the fastest growing goods state and the slowest diminishing market state.

In the example above, the powers of T have a simple meaning: $T^k(x)$ consists of the states into which x could be transformed in k time units. It seems worthwhile therefore to study the sequence

$$C, T(C), T^2(C), \dots, T^k(C), \dots \tag{6.9}$$

where C is a non-singular monotone set of the same type as T. In particular, we call C an *eigenset* of T if

$$T(C) = \lambda C \quad \text{for some } \lambda > 0 . \tag{6.10}$$

Then (6.9) consists of multiples of C. Eigensets of T^* also provide information about (6.9). If D^* is such an eigenset,

$$\langle \lambda^{-k}T^k(C), D^* \rangle = \langle C, \lambda^{-k}T^{*k}(D^*) \rangle = \langle C, D^* \rangle$$

for every positive integer k. Assuming T is of concave type, we see that the supremum of the monotone gauge $\langle \cdot, D^* \rangle$ is the same on each of the monotone sets $\lambda^{-k}T^k(C)$, so that the levels of $\langle \cdot, D^* \rangle$ are "orbits" along which $\lambda^{-1}T$ acts.

In view of the strong property just described, the existence and uniqueness of eigensets is an important matter. We shall call T a *primary* monotone process if both T and T^* have (non-singular) eigensets. It may be shown that then only one value of λ can be involved, in fact $\lambda = \underline{\lambda} = \bar{\lambda}$. Hence a primary monotone process is necessarily evenly growing. The following partial converse holds, too. Assume that T is evenly growing, and that there exists vectors $x \gg 0$ and $x^* \gg 0$ such that $\lambda x \in T(x)$ and $\lambda x^* \in T^*(x^*)$, where $\lambda = \underline{\lambda} = \bar{\lambda}$. (Ordinarily, "evenly growing" only implies the existence of some such $x > 0$ and $x^* > 0$.) Then T is primary. The hypothesis that $x \gg 0$ and $x^* \gg 0$ may be weakened still further: taking T to be of concave type for definiteness, one need only assume that $T^k(x)$ and $(T^{*k})^{-1}(x^*)$ are non-singular monotone sets (of concave type) for k sufficiently large.

The class of "simple" monotone processes also arises in this context. Let T be of concave type. Let P be any *sub-orthant* of P_n, i.e. a closed face of P_n as a polyhedral convex set in R^n. Then

$$\bigcup_{x \in P} T(x) \quad \text{and} \quad \bigcap_{z \gg 0} \bigcup_{x \in P} T(z+x) \tag{6.11}$$

are sub-orthants of P_n, too. If the first of these coincides with P, we say P is *self-reproducing* under T; if the second coincides with P, we say P is *asymptotically self-reproducing* under T. (Incidentally, P is self-reproducing under T if and only if its annihilator sub-orthant P^* is asymptotically self-reproducing under the polar of T, and dually.) We call T a *simple* monotone process if no sub-orthant, other than $\{0\}$ or P_n itself, is either self-reproducing or asymptotically self-reproducing under T. (A monotone process of convex type is called simple if its adjoint is simple.) The main result about simple processes is that they are all primary (and hence evenly growing). As an example: *every positive process is a simple primary monotone process.*

We have just given a number of conditions which guarantee the existence of eigensets. The uniqueness of such sets (up to positive scalar multiples) seems to be a far more difficult question. For instance, non-uniqueness occurs even for certain *positive idempotent* ($T^2 = T$) monotone processes.

Although we have not found easy criteria for "uniqueness", we can prove the equivalence of the uniqueness property and a very significant kind of behavior of T^k in the limit. Suppose that T is of concave type, and that $\bar{C}$ and $\bar{D}^*$ are non-singular eigensets of T and T^*, respectively, normalized so that $\langle \bar{C}, \bar{D}^* \rangle = 1$. Define T_0 by

$$T_0(x) = \langle x, \bar{D}^* \rangle \bar{C} . \tag{6.12}$$

Then T_0 is an *idempotent* monotone process of concave type such that

$$T_0 T = \lambda T_0 \quad \text{where } \lambda = \underline{\lambda} = \bar{\lambda} . \tag{6.13}$$

The theorem is that the uniqueness of $\bar{C}$ and $\bar{D}^*$ as eigensets (up to positive multiples) is a necessary and sufficient condition in order that the powers of $\lambda^{-1}T$ converge to T_0, in the sense that

$$\lambda^{-k} T^k(C) \to T_0(C) \quad \text{as } k \to \infty \tag{6.14}$$

(uniformly, as compact convex sets in R^n) for each non-singular monotone set C of concave type. This theorem is quite hard to prove; perhaps the theory of topological semi-groups could be used to advantage here.

Other interesting theorems about limiting behavior can be developed around a notion of "norm". Fix any two non-singular monotone sets C_0 and C_0^* of concave type, polar to each other, in P_n and P_n^*, respectively. Write $\|x\|$ for $\langle x, C_0^* \rangle$ and $\|x^*\|$ for $\langle C_0, x^* \rangle$. Define

$$\|C\| = \sup \{ \|x\| \mid x \in C \} \tag{6.15a}$$

if C is a monotone set of concave type, but

$$\|C\| = \inf \{ \|x\| \mid x \in C \} \tag{6.15b}$$

if C is of convex type. Finally, define

$$\|T\| = \sup \{ \|T(x)\| / \|x\| \mid x > 0 \} . \tag{6.16}$$

These norms satisfy

$$\|T(C)\| \leqq \|T\| \cdot \|C\|, \quad \langle C, D^* \rangle \leqq \|C\| \cdot \|D^*\| , \tag{6.17}$$

and other respectable rules. Surprisingly enough,

$$\|T^*\| = \|T\| . \tag{6.18}$$

The most striking result is an analog of the "*spectral norm theorem*":

$$\lim_{k\to\infty} \|T^k\|^{1/k} = \bar{\lambda} . \tag{6.19}$$

When T is evenly growing, one also has

$$\lim_{k\to\infty} \langle T^k(x), x^*\rangle^{1/k} = \bar{\lambda} = \underline{\lambda} \tag{6.20}$$

for any $x \gg 0$ and $x^* \gg 0$. In fact then

$$\lim_{k\to\infty} \langle T^k(C), D^*\rangle^{1/k} = \bar{\lambda} = \underline{\lambda} , \tag{6.21}$$

provided the monotone sets C and D^* are non-singular.

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