The Elementary Vectors of a Subspace of R^N

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1. INTRODUCTION

This paper concerns some connections between convex analysis, network flows and matroid theory.

Let K be an arbitrary subspace of R^N, where R is the real number system. Regarding K as a chain group in the sense of Tutte, one can pass to the corresponding matroid. Combinatorial facts deduced from general matroid theory may then be reinterpreted in terms of the original vectors in K. The results so obtained reflect the fact that K is not just a real vector space, but has further structure because of its particular disposition within R^N. Specifically, the matroid analysis of K deals with the way K intersects the special subspaces of R^N spanned by the canonical coordinate axes. Now, the natural ordering of R allows one to enlarge the finite category of intersections under scrutiny to include closed "orthants", and in general all the polyhedral convex cones generated by the various positive and negative halves of coordinate axes. The

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combinatorial theory of such intersections is what we want to consider here.

One motivation is the question of "orientation" in matroids. In the elementary cycles and cocycles of a directed linear graph, there is a natural sense in which two elements have the same or the opposite orientation. There are certain combinatorial results, such as Minty's "colored arc lemma", which involve orientations but are otherwise really assertions about matroids. Minty has recently presented in [8] an interesting development of matroid theory, in which certain orientations are introduced axiomatically in terms of "digraphoids", so that abstract generalizations of the graph results are true. A "digraphoid", he has shown, corresponds to a dual pair of matroids which are "regular" in Tutte's terminology. There does exist, then, an abstract theory of "oriented" or "signed" matroids which implies that the matroids involved are regular.

A much broader theory of orientation ought to be possible, in our opinion. Regular matroids arise from subspaces of R^N, but only subspaces of an extremely special type. For any subspace K of R^N, however, there is a natural way of using the signs of the coordinates of the vectors to introduce orientations into the corresponding matroid. The study of the signed matroid amounts to the generalized intersection problem posed above. We shall demonstrate that, for such signed matroids, several theorems are valid which are far from obvious, and which even have important well-known non-matroid theorems as consequences.

No attempt is made here to develop a theory of signed matroids axiomatically. We are concerned, rather, with showing that there are interesting and significant examples which any such theory ought to encompass.

The paper is partly expository, in that we also aim to describe a certain bridge between results in convex analysis and graph theory. Well-known theorems about systems of linear inequalities can sometimes be reformulated as seemingly much simpler combinatorial theorems about the way a subspace K intersects some orthant. This is true of the duality theorem for linear programs, as has been pointed out by Tucker. We want to show that, in this form, the theorems about inequalities correspond to other well-known theorems of a combinatorial character about graphs, which have been arrived at by an entirely different route. The idea is to specialize K to a space of network flows. It turns out, for instance, that Minty's "colored arc lemma" is essentially a special case of the classical lemma of Farkas.

The result to which we would most like to draw the reader's attention is Theorem 3, a versatile existence theorem which extends Minty's theorem for "interval networks" in [6]. It is a partly combinatorial result about the consistency of systems of linear inequalities, suited in particular for application to the dual convex programs in [11].

Note. Since this paper was submitted, P. Camion has informed us that a more general form of Theorem 3 is proved in his unpublished thesis [23] in terms of modules over totally ordered integral domains. The thesis also contains results equivalent to Theorems 1 and 6, which we display below as corollaries of basic theorems in convex analysis, and there are ideas similar to those in Section 7 about using the simplex algorithm to determine which of the alternatives in Theorem 6 holds in a given case. Some results from Camion's thesis are stated without proof in an appendix to [22].

2. ELEMENTARY VECTORS AND SUPPORTS

It will be helpful to think of the vectors $X = (x_1, \dots, x_N)$ in R^N as real-valued functions on a certain finite set $E = \{e_1, \dots, e_N\}$, with $X(e_i) = x_i$. The *support* of X is then a certain subset of E, namely the set of e_i's such that $x_i \neq 0$. An *elementary vector* of K is defined to be a non-zero vector of K whose support is minimal, i.e. does not properly contain the support of any other non-zero vector of K. The system of subsets of E consisting of the supports of the elementary vectors of K (which we call the *elementary supports of* K) is, of course, the matroid associated with K.

It is important to keep in mind that two elementary vectors X and X' of K having the same support have to be scalar multiples of each other. Indeed, if $\lambda \in R$ is chosen so that one of the non-zero components of λX equals the corresponding component of X', then $X' - \lambda X$ is a vector of K whose support is properly smaller, so that $X' - \lambda X = 0$. Hence K has only *finitely many elementary vectors, up to scalar multiples.* The ratios between the components of an elementary vector do not depend on the arbitrary multiple. Thus a certain finite "ratio system" is uniquely and intrinsically defined by K. This "ratio system" determines K completely, because K is the subspace generated by its elementary vectors (see Section 3). An interesting "combinatorial" problem is to determine necessary and sufficient conditions on a "ratio system" in order that it arises in this fashion. This is closely related to the problem of characterizing the classes of real matrices combinatorially equivalent to each other in the sense of Tucker (see Section 6). Most of the results below concern necessary conditions on the patterns of signs of the ratios or matrix elements.

By a *signed set* in E, we shall mean a subset S which has been partitioned into two further subsets S^+ and S^- (possibly empty). We shall say that S contains an element e_i *positively* or *negatively*, according to whether $e_i \in S^+$ or $e_i \in S^-$. A signed subset can be represented in an obvious manner by an N-vector formed from the symbols $+$, $-$ and 0.

With each vector X of R^N, we associate a signed set S formed from the support of X, where S^+ consists of the elements e_i with $x_i > 0$, and S^- consists of the elements e_i with $x_i < 0$. We call this the *signed support* of X. A signed set which is the signed support of some vector in K is said to be a *signed support of* K. It is *elementary* if it actually comes from an elementary vector of K.

The system of elementary signed supports of K may be regarded as a sort of "signed matroid." Its properties include an extensive duality with the system of elementary signed supports of $K^\perp$, the orthogonal complement of K, as one would readily expect from ordinary matroid theory.

A special case to which we shall often appeal for motivation, and which therefore deserves a brief review, is the case where E is the set of arcs of a directed graph. Here we interpret the vectors in K as *circulations* in the graph, i.e. flows which are conservative at every vertex. Thus $X \in K$ if and only if X is orthogonal to every row of the (vertex vs. arc, signed) incidence matrix of the graph. For the general theory of such flows, we refer the reader to the exposition of Berge [2].

The *elementary* circulations are easy to determine. Given any elementary cycle S in the graph and a real number α, a "circulation of intensity α around S" is obtained by setting $x_i = \alpha$ if S contains e_i in the sense of its orientation, $x_i = -\alpha$ if S contains e_i in the opposite sense, and $x_i = 0$ if S does not contain e_i at all. On the other hand, a simple argument invoking the conservation condition at each vertex shows that every non-zero circulation contains some cycle in its support. It follows that the elementary vectors for this choice of K are precisely the circulations of non-zero intensity around elementary cycles. The elementary signed supports of K can be identified with the elementary cycles themselves.

In this example, $K^\perp$ is the subspace generated by the rows of the incidence matrix. Thus $Y \in K^\perp$ means that Y is a *tension* in the graph, i.e. that there exists some "potential" function on the vertices of the graph such that each y_i is obtained by subtracting the potential at the initial vertex of e_i from the potential at the final vertex of e_i. Given any elementary cocycle S in the graph and a real number α, one can construct a "tension of intensity α across S", much as above. The non-zero tensions of this form turn out to be precisely the elementary vectors of $K^\perp$, so that the elementary signed supports of $K^\perp$ are the elementary cocycles.

Much of the theory of linear inequalities, our other main source of motivation, concerns "linear systems of variables" rather than subspaces of R^N. But the two settings are really interchangeable. In the "linear variables" case, one deals with the pairs of vectors $\boldsymbol{U} \in R^m$ and $\boldsymbol{V} \in R^n$ satisfying $\boldsymbol{U}\boldsymbol{A} = \boldsymbol{V}$, where $\boldsymbol{A}$ is a given $m \times n$ matrix. The set of such pairs $\boldsymbol{X} = (\boldsymbol{U}, \boldsymbol{V})$ forms, of course, a certain subspace K of R^N, with $N = m + n$. The orthogonal complement $K^\perp$ of this K consists of the pairs $\boldsymbol{Y} = (\boldsymbol{U}', \boldsymbol{V}')$ such that $\boldsymbol{U}' \in R^m$, $\boldsymbol{V}' \in R^n$, and $\boldsymbol{V}'\boldsymbol{A}^T = -\boldsymbol{U}'$, where $\boldsymbol{A}^T$ is the transpose of $\boldsymbol{A}$. Tucker's theory of combinatorial equivalence tells us how to represent an arbitrary subspace K in this way by various matrices $\boldsymbol{A}$. More will be said about this in Section 6.

Everything that follows would still be valid if R were replaced by *any* ordered field.

3. HARMONIOUS SUPERPOSITION

A known result about circulations in directed graphs is that every such circulation $\boldsymbol{X}$ can be represented (non-uniquely) as a superposition $\boldsymbol{X}_1 + \ldots + \boldsymbol{X}_r$, where each $\boldsymbol{X}_k$ is a circulation around an elementary cycle of the graph. Moreover, the cycles can be chosen so that the orientations of their arcs agree with the signs of the corresponding flow components in $\boldsymbol{X}$, see [2, p. 145]. In particular, the support of $\boldsymbol{X}$ is then the union of the elementary cycles involved. This theorem can be generalized to arbitrary K, as we now show.

Let us say that two vectors X and X' in R^N are *dissonant*, if, for some i, the components x_i and x'_i are non-zero and opposite in sign. Thus X and X' are *in harmony* (i.e. fail to be dissonant) if and only if $x_i x'_i \geq 0$ for every i.

Theorem 1. *Let X be any non-zero vector in K. Then there exist elementary vectors $X_1, \ldots, X_r$ of K, such that $X = X_1 + \ldots + X_r$. These elementary vectors may be chosen such that each is in harmony with X and has its support contained in the support of X, but none has its support contained in the union of the supports of the others, and such that r does not exceed the dimension of K or the number of elements in the support of X.*

Proof. The conditions on r follow immediately from the conditions on the supports of $X_1, \ldots, X_r$, and they need not be mentioned further. It suffices to treat the theorem in the notationally simpler case where $X \geq 0$, i.e. $x_i \geq 0$ for all i. We must show that X can be expressed as the sum of non-negative elementary vectors of K, each of which has an element in its support not belonging to the support of any of the others. A preliminary step is to show that there exists at least one non-negative elementary vector whose support is contained in the support of X. Assume inductively that this fact has already been established for all non-zero non-negative vectors $X' \in K$ whose supports are properly smaller than that of X. Let X_0 be any elementary vector of K (not necessarily non-negative) whose support is contained in the support of X. Replacing X_0 by its negative if necessary, we can assume that X_0 has a positive component. Then there exists a largest positive scalar λ such that $\lambda X_0 \leq X$. If $\lambda X_0 = X$, X is itself a non-negative elementary vector. Otherwise, $X' = X - \lambda X_0$ is a non-negative vector of K whose support is contained in the support of X but does not contain the support of X_0. By induction, there exists a non-negative elementary vector of K whose support is contained in the support of X', and hence in the support of X. We can proceed now to prove the theorem itself in the same way. Assume inductively that the theorem has already been established for all non-zero non-negative vectors $X' \in K$ whose supports are properly smaller than the support of X. Repeat the argument above, but this time taking $X_0 \geq 0$, as has just been shown possible. The induction hypothesis yields a decomposition

$$X_2 + \ldots + X_r = X' = X - \lambda X_0 .$$

Setting $X_1 = \lambda X_0$, we get the desired decomposition of X.

Corollary. *The elementary vectors of K generate K algebraically.*

Theorem 1 has been depicted as an extension of a result about graphs, but it is actually equivalent to a fundamental theorem in convex analysis. The theorem in question says that each non-zero vector in a polyhedral convex cone containing no whole lines may be expressed as a sum of r extreme vectors of the cone, where r need not exceed the dimension of the face of the cone in which the given vector lies.

It is not hard to deduce Theorem 1 from this cone theorem. One argues that the set of non-negative vectors of K is a polyhedral convex cone K_+ containing no whole lines, whose extreme vectors are elementary. The faces of K_+ correspond to the "non-negative" signed supports of K. It is just as easy, on the other hand, to deduce the cone theorem from Theorem 1. This is even a convenient route for attaining various important facts about polyhedral convex cones, since the direct proof furnished above for Theorem 1 is so elementary. Recall that, by definition, a polyhedral convex cone C in R^m can be represented as the inverse image of the non-negative orthant of some R^N under some linear transformation T. If C contains no whole lines, T is one-to-one from R^m onto a certain subspace K (the range space of T), and T carries C onto K_+. Application of Theorem 1 to K_+ yields the facts about C.

The study of signed sets is greatly aided by Theorem 1. We can define, in the obvious parallel way, what we mean by two *signed sets* being *dissonant* or *in harmony*. If $S_1, \ldots, S_r$ are signed sets pairwise in harmony, a new signed set S, the *harmonious union* of $S_1, \ldots, S_r$, can be formed by taking

$$S^+ = S_1^+ \cup \ldots \cup S_r^+ \quad \text{and} \quad S^- = S_1^- \cup \ldots \cup S_r^-$$

(In the dissonant case, this S^+ and S^- would overlap, so that there would be no natural way of introducing signs in the union.) If vectors $X_1, \ldots, X_r$ are pairwise in harmony, so are their signed supports, and vice versa. The harmonious union of these signed supports is then the signed support of $X_1 + \ldots + X_r$. Theorem 1 immediately yields the following result, according to which the properties of the signed supports of K can entirely be deduced from those of the elementary signed supports.

Theorem 2. *Every signed support of K is a harmonious union of elementary signed supports of K. On the other hand, every such harmonious union is a signed support of K.*

4. FUNDAMENTAL EXISTENCE THEOREM

In applications of flow theory, the question often comes up as to whether there exists a circulation X whose components x_i lie within certain given ranges I_i depending on the arcs e_i. It may be required for some arcs, say, that $0 \leq x_i \leq k_i$, where k_i is the "capacity" of the arc, while for other arcs x_i is to assume a constant value specified in advance. Some existence theorems pertaining to closed intervals, for instance, are presented by Berge [2, p. 157–160]. These are all really special cases of a theorem of Minty [6] for arbitrary intervals (i.e. non-empty connected sets of real numbers, not necessarily closed or open or bounded, possibly degenerating to a single point).

We shall now prove that Minty's theorem is valid for arbitrary K, if reformulated in terms of elementary vectors.

Theorem 3. *Let $I_1, \ldots, I_N$ be arbitrary real intervals. Then one of the following alternatives holds, but not both:*

(a) *There exists a vector X of K such that $x_i \in I_i$ for $i = 1, \ldots, N$;*

(b) *There exists an elementary vector Y of $K^\perp$ such that $y_1 I_1 + \ldots + y_N I_N > 0$ (i.e. the interval obtained by letting $y_1 x_1 + \ldots + y_N x_N$ vary over all choices of $x_i \in I_i$ lies entirely to the right of 0).*

Proof. The conditions are mutually exclusive, because $y_1 x_1 + \ldots + y_N x_N > 0$ is impossible when $X \in K$ and $Y \in K^\perp$. Let Q be the set of all vectors $X \in R^N$ such that $x_i \in I_i$ for $i = 1, \ldots, N$. In the terminology of [10], Q is a partial polyhedral convex set. If condition (a) fails, Q does not meet K, and a certain separation theorem of the writer [10] may be applied.

This gives the existence of a vector $Y \in K^{\perp}$, such that $y_1 x_1 + \ldots + y_N x_N > 0$ for every $X \in Q$, i.e. $y_1 I_1 + \ldots + y_N I_N > 0$. We must demonstrate that this Y can actually be replaced by an *elementary* vector of $K^{\perp}$. Theorem 1 allows us to set $Y = Y_1 + \ldots + Y_r$, where the vectors $Y_j = (y_{j1}, \ldots, y_{jN})$ are elementary vectors of $K^{\perp}$ pairwise in harmony with each other. The distributive law $(\lambda_1 + \lambda_2)I = \lambda_1 I + \lambda_2 I$ holds for any interval I provided $\lambda_1 \lambda_2 \geq 0$. Therefore

$$y_1 I_1 + \ldots + y_N I_N = \sum_{j=1}^{r} (y_{j1} I_1 + \ldots + y_{jN} I_N)$$

by "harmony." The interval represented on the left lies wholely in the positive part of R, so the same must be true of one of the r intervals corresponding to $Y_1, \ldots, Y_r$ on the right. (If all r intervals contained a non-positive number, then so would their sum.) Thus

$$y_{j1} I_1 + \ldots + y_{jN} I_N > 0$$

for some elementary vector Y_j of $K^{\perp}$, which is what was to be proved.

Notice that (b) in Theorem 3 is a combinatorial condition, in that there are essentially only *finitely* many possibilities to test. Up to positive multiples, $K^{\perp}$ has only finitely many elementary vectors, and a positive multiple of Y makes no difference in (b). In the graph example, the elementary vectors of $K^{\perp}$ correspond to cocycles, and the multiple can always be chosen so that all the components y_i of Y are $+1$, -1 or 0. Then condition (a) holds if and only if, for every elementary cocycle of the graph,

$$0 \in y_1 I_1 + \ldots + y_N I_N = \sum\nolimits_1 I_i - \sum\nolimits_2 I_i ,$$

where $\sum_1$ is the sum over the indices i such that the given cocycle contains the arc e_i in the direction of its orientation, and $\sum_2$ is the sum over the indices such that e_i is contained in the opposite direction. The "max-flow-min-cut" theorem is readily deduced from this, as has been explained by Minty.

Theorem 3 has been derived from a separation theorem of convex analysis which is stronger than the well-known lemma of Farkas. Actually, this separation theorem can be derived in turn from Theorem 3, using the fact that, by definition, every partial polyhedral convex set is the inverse image under some linear transformation of a set of "paralellopiped form"

$$\{x | x_i \in I_i \text{ for every } i\},$$

where the I_i are intervals.

Existence alternatives for inequalities involving the variables in a "linear system" $UA = V$ can be obtained by applying Theorem 3 to the subspace K described at the end of the Section 2. Tucker's results in [13] can be established this way. Some special cases will be considered below.

As an immediate combinatorial application of Theorem 3, we shall show how the signed supports of K may be constructed directly from those of $K^\perp$. From matroid theory it is known, of course, how to construct the elementary supports if signs are disregarded. One takes the collection of non-empty subsets S of E such that no elementary support of K meets S in just a single element; the minimal sets among these are the elementary supports of $K^\perp$. The following theorem shows what modification works for the *signed* supports.

Theorem 4. *Let S be a signed set in E. In order that S be a signed support of $K^\perp$, it is necessary and sufficient that every elementary signed support of K not disjoint from S be dissonant with S.*

Proof. The necessity is on the surface. For if $X \in K$ and $Y \in K^\perp$ had signed supports in harmony and not disjoint, then $x_i y_i \geq 0$ for every i with strict inequality for at least one i, contradicting $x_1 y_1 + \ldots + x_N y_N = 0$. To prove the sufficiency, we apply Theorem 3, with the roles of K and $K^\perp$ reversed, to the case where $I_i = (0, +\infty)$ for $e_i \in S^+$, $I_i = (-\infty, 0)$ for $e_i \in S^-$, and $I_i = \{0\}$ for $e_i \in S$. If S is not a signed support of K, that means there is no $Y \in K^\perp$ such that $y_i \in I_i$ for every i. Then by Theorem 3 there exists an elementary vector $X \in K$, such that

$$x_1 I_1 + \ldots + x_N I_N > 0 .$$

This implies that $x_i \geq 0$ for $e_i \in S^+$ and $x_i \leq 0$ for $e_i \in S^-$, with strict inequality for at least one $e_i \notin S$. The signed support of X is then an elementary signed support of K in harmony with S, but not disjoint from S.

5. PAINTINGS

Certain combinatorial problems in graphs involve a specified partitioning of the set of arcs into several subsets. A happy way of describing the partitioning, which has been exploited by Minty, is to say that the arcs have been "painted" various colors. One can then speak of a "black and red cocycle", meaning a cocycle constructed exclusively of "black" arcs and "red" arcs, and so forth. (A black and red cocycle could be entirely black or entirely red.)

Here we shall present several results about the existence of signed supports matching a given "painting." The first is a complementarity theorem.

Theorem 5. *Let each of the elements e_i of E arbitrarily be painted white, green or red (where any of the colors can remain unused). Then there exist a green and white signed support S of K and a red and white signed support S' of $K^\perp$, such that S and S' have no element in common, but every white element is contained in S positively or in S' positively."*

Proof. From among the vectors $X \in K$ such that $x_i \geq 0$ for e_i white and $x_i = 0$ for e_i red, choose one whose support contains a maximal number of white elements. Call it X_0, and let S be its support. Take $I_i = (0, +\infty)$ for e_i white and not in the support of X_0, $I_i = (-\infty, +\infty)$ for e_i red, and $I_i = \{0\}$ for every other i. If there exists a vector $Y \in K^\perp$ such that $y_i \in I_i$ for every i, the support S' of Y, along with S, meets the requirements of the theorem. Suppose, therefore, that no such Y exists. We shall show that leads to a contradiction. By Theorem 3 (with K and $K^\perp$ reversed), there alternatively exists some $X \in K$, such that

$$x_1 I_1 + \ldots + x_N I_N > 0 .$$

The choice of intervals forces $x_i = 0$ for e_i red and $x_i \geq 0$ for e_i white and not in the support of X_0, with $x_i > 0$ for at least one of the latter elements. Then $X + \lambda X_0$, for λ positive and sufficiently large, has a green and white signed support containing no white element negatively and containing at least one more white element than was the case with X_0. This conflicts with the maximality in the selection of X_0.

Corollary. *There exist non-negative vectors $X \in K$ and $Y \in K^\perp$ which are complementary, i.e. such that $x_i y_i = 0$ and $x_i + y_i > 0$ for every i.*

Proof. Paint every element white.

This corollary is a well-known complementary slackness theorem of Tucker [13], which can be made the cornerstone of linear programming theory. Theorem 5 itself could be deduced without much trouble from Tucker's many results about dual linear systems of variables in [13], so only its formulation here, as a combinatorial theorem concerning dual systems of signed sets, is really new. The interesting thing about this formulation, however, is that it leads quickly to the following generalization of Minty's fundamental "colored arc lemma" [6] for directed graphs.

Theorem 6. *Let one of the elements e_i of E be painted black, and let each of the other elements arbitrarily be painted white, green or red. Then one of the following alternatives holds, but not both:*

(a) *There exists an elementary signed support of K containning the black element and otherwise only green and white elements, with the black and white elements contained positively;*

(b) *There exists an elementary signed support of $K^{\perp}$ containing the black element and otherwise only red and white elements, with the black and white elements contained positively.*

Proof. If both conditions could be satisfied simultaneously, one would have overlapping signed supports of K and $K^{\perp}$ in harmony, contrary to Theorem 4. Thus (a) and (b) are mutually exclusive. On the other hand, suppose the black element is repainted white and apply Theorem 5. The S obtained can be expressed as a harmonious union of green and white elementary signed supports of K by Theorem 2, and similarly for S' with "red" in place of green. The previously black element belongs to either S or S' and hence to one of the elementary signed supports in these decompositions. That signed support satisfies either (a) or (b).

Corollary. *Each element of E belongs either to some non-negative (i.e. $S^+ = S$, $S^- = \phi$) elementary signed support of K or to some non-negative elementary signed support of $K^{\perp}$, but not both.*

Proof. Paint the element in question black and every other element of E white.

In the directed graph case, the corollary reduces to the fact that every arc belongs either to some "unidirectional" elementary cycle or to some "unidirectional" elementary cocycle.

Minty has demonstrated in [8] that the simpler "unsigned" version of the property in Theorem 6, namely in which one omits the color white and all mention of signs, may be adopted as a fundamental axiom of matroid theory. He has not developed the signed version as an axiom, although he has shown it is valid for his "digraphoids". According to Theorem 6, the signed version is actually valid for a much broader class of systems than "digraphoids."

An important virtue of the "colored arc lemma" in Minty's convex programming theory for monotone networks [6] is that an efficient combinatorial algorithm actually constructs an elementary cycle or cocycle satisfying alternative (a) or (b). This prompts one to ask whether a constructive procedure exists for the more general case of Theorem 6, too. The proof we have given here is not constructive. We shall see below, however, that the construction can be effected by the simplex algorithm of linear programming.

6. MATRIX REPRESENTATIONS

The relationship between Tucker's combinatorial theory for linear systems of variables, "digraphoids," and the study of elementary vectors and signed supports will now be explained. The results described below are all known, in one way or another, but they need to be worked up together in a certain way as preparation for their use in the next section.

Suppose that, for a certain $m \times n$ matrix $A = (a_{ij})$, K is given by $UA = V$ as at the end of Section 2. The vectors X in K are then precisely the ones whose components satisfy

$$\sum_{i=1}^{m} x_i a_{ij} = x_{m+j} \quad \text{for} \quad j = 1, \ldots, n.$$

Here the values of $x_1, \ldots, x_m$ can be specified arbitrarity, and the values of the remaining components $x_{m+1}, \ldots, x_{m+n} = x_N$ are then explicitly given. At the same time, the vectors Y in $K^\perp$ are precisely the ones whose components satisfy

$$\sum_{j=1}^{n} a_{ij} y_{m+j} = - y_i \quad \text{for} \quad i = 1, \ldots, m.$$

These dual systems of equations may conveniently by summarized in a tableau.

$$\begin{array}{cccccc} & y_{m+1} & \cdots & y_{m+n} & & \\ x_1 & a_{11} & \cdots & a_{1n} & = & -y_1 \\ \vdots & \vdots & & \vdots & & \vdots \\ x_m & a_{m1} & \cdots & a_{mn} & = & -y_m \\ & = x_{m+1} & & = x_{m+n} & & \end{array}$$

We shall call such a tableau a *Tucker representation* of the subspaces K and $K^{\perp}$. For notational simplicity, we have only pictured a representation in which the symbols $x_1, \ldots, x_N$ occur in undisturbed order along the margins of the tableau. In reality, of course, there will usually be numerous representations, involving different arrangements of the symbols. Every such representation entails the partitioning of E into two subsets D and D', such that the components x_i of a vector $\boldsymbol{X}$ in K for e_i in D are uniquely determined by the components for e_i in D', while the latter components take on all possible combinations of values as $\boldsymbol{X}$ ranges over K. With respect to $K^{\perp}$, D and D' have the opposite property.

Tableaus which represent the same complementary pair of subspaces are said to be *combinatorially equivalent* (along with their corresponding matrices $\boldsymbol{A}$). How to pass arithmetically from any given tableau to any other combinatorially equivalent tableau has been thoroughly clarified by Tucker [14, 15, 16]. "Pivoting" and rearranging are all that is required. A simple pivot step corresponds to a classical elimination procedure for the dual systems of equations. Any non-zero entry in the tableau may be selected as "pivot"; one then passes to an adjacent representation, in which D and D' are modified by interchanging the e_i of the pivot row with the e_i of the pivot column. As far as getting an initial representation is concerned, that is a very easy matter, at least if K is defined as the subspace orthogonal to a known finite set of vectors in R^N, or as the subspace generated by such a set. (That is the situation in the graph example.)

Tucker's theory grew out of studies of the simplex algorithm for linear programs. But it is also relevant to some ideas Tutte has exploited for representing matroids, as we shall now relate.

Thinking of the vectors $\boldsymbol{X}$ in K as functions on E, we may restrict them to a given subset D of E. The restrictions may be viewed as vectors in R^M, where M is the number of elements in D. The Tucker representation corresponds to the case where the restriction mapping (a linear transformation) is one-to-one from K onto R^M. The mapping clearly is "one-to-one" if and only if no non-zero vector of K has its support disjoint from D. It is "onto" if and only if no non-zero vector of $K^{\perp}$ has its support contained in D. Indeed, in these conditions it is enough to speak of elementary vectors. The case where both conditions hold is where D is minimal with respect to the property that it meets every elementary support of K, or equivalently where D is maximal with respect to the property that it contains no elementary support of $K^{\perp}$.

A set D with the latter properties is called a *dendroid* of K by Tutte. Notice that the complement of a dendroid of K is a dendroid of $K^{\perp}$. In the example of a connected directed graph, of course, the dendroids of K are the sets of arcs maximal with respect to the property that their deletion would not disconnect the graph; the dendroids of $K^{\perp}$ are the maximal trees of the graph. In general, according to the analysis above, *the various partionings of $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ into "row symbols" and "column symbols" in the Tucker representations of K and $K^{\perp}$ correspond to the possible ways of partitioning E into a dendroid D of K and a dendroid D' of $K^{\perp}$.*

Given a Tucker representation of K and $K^{\perp}$ in the notationally simple form above, the $m \times N$ matrix $[\boldsymbol{I}_m, \boldsymbol{A}]$ (where $\boldsymbol{I}_m$ is the $m \times m$ identity matrix) is what Tutte calls a *standard representative matrix* for K (and its matroid). The rows of this matrix are evidently *elementary* vectors of K forming a basis of K. Likewise, $[-\boldsymbol{A}^T, \boldsymbol{I}_n]$ is a standard representative matrix for $K^{\perp}$ (and the dual matroid), and its rows are elementary vectors of $K^{\perp}$ forming a basis of $K^{\perp}$.

Two such standard representative matrices are implicit similarly in a general Tucker representation. They are obtained by applying to the columns of $[\boldsymbol{I}_m, \boldsymbol{A}]$ and $[-\boldsymbol{A}^T, \boldsymbol{I}_n]$ the permutation which is required to restore the symbols x_i from the order in which they occur, down the left side and across the bottom of the tableau, to the order $x_1, \ldots, x_N$. *Every Tucker representation thus yields a basis of elementary vectors for K and one for $K^{\perp}$.* The bases so obtained will be called *elementary bases.* (A basis consisting of elementary vectors is not actually an elementary basis unless one can also "select an identity matrix from the components.")

In matroid terms, a dendroid D of K yields a certain unique family of elementary supports of K (namely, those of the vectors in the corresponding elementary basis), each having exactly one element e_i in common with D. From matroid theory, it is known that, for any elementary support S of K and any $e_i \in S$, one can find a dendroid D giving rise this way to S and having e_i as its only element in common with S. We can state that result equivalently as follows: each elementary vector of K having a component equal to 1 belongs to some elementary basis of K, and therefore occurs in some Tucker representation. *Tucker's "pivoting" formulas thus serve to compute all the elementary vectors of K and $K^\perp$, up to scalar multiples.*

In a directed graph, for example, an elementary vector of K having some component equal to 1 is a circulation of intensity 1 around some elementary cycle; hence it is actually a representative vector for some elementary cycle, and all its components equal $+1$, -1 or 0. The matrices in the Tucker representations thus must have all their components equal to $+1$, -1 or 0. If the graph is connected, each Tucker representation corresponds to a certain maximal tree D' of E. The elementary basis of K which can be read from the tableau gives the fundamental basis of elementary cycles associated with the tree D'. Pivoting in the tableau is then an arithmetic expression of the purely combinatorial operation of passage to an adjacent tree. That is why the general algorithms of linear programming can be supplemented by simpler combinatorial algorithms, when network problems are involved; see Dantzig's comments [3, Chapter 17].

More generally, a simplified combinatorial approach with strong graph-theoretic analogies is possible in the context of Minty's "digraphoids" and "unimodularity." A matrix $\boldsymbol{A}$ is said to have the unimodular property, if every square submatrix of $\boldsymbol{A}$ has determinant equal to $+1$, -1 or 0. Actually, by Tucker's theory, this is equivalent to the property that every matrix combinatorially equivalent to $\boldsymbol{A}$ (including $\boldsymbol{A}$ itself) have only $+1$'s, -1's and 0's as components. The latter property would make a better definition of unimodularity, in the author's opinion, since it is the property that one is directly concerned with in linear programming applications. (If an initial linear programming tableau in Tucker's format has integral "margins", and if its "non-marginal" matrix has the unimodular property, then the arithmetic of the simplex algorithm will be trivial, and the solutions calculated will be integral.) According to the above, any *circulation matrix* of a directed graph (i.e., the tableau matrix $\boldsymbol{A}$ of some Tucker representation of the space K of circulations in a directed graph) has the unimodular property. The derivation given here for this well-known and important result hinged merely on the fact that, for graphs, every elementary vector of K is a multiple of a *primitive* vector, i.e. a vector having every component equal to $+1$, -1 or 0. In general, let us call a subspace K with the latter property a *unimodular* subspace of R^N. We can say then that K is unimodular if and only if the matrices $\boldsymbol{A}$ in its Tucker representations have the unimodular property. *The study of matrices with the unimodular property is thus equivalent to the study of certain subspaces of* R^N *and their elementary vectors.* Such unimodular subspaces are what Tutte would call "regular chain groups over the real numbers." Minty has shown [8, Appendix A] that the systems of elementary signed supports of such subspaces and their orthogonal complements are precisely the objects of his "digraphoid" theory. Minty's results may therefore be regarded as a contribution to the theory of matrices with the unimodular property, in which everything is built up axiomatically in analogy with graphs.

The class of matrices with the unimodular property is, of course, closed under many operations besides those of Tucker's combinatorial equivalence (pivoting, and permutation of rows and columns), notably the operations of

(a) taking submatrices;
(d) multiplying various rows or columns through by -1;
(c) taking transposes;
(b) appending a new row or column having only one non-zero component, and that a $+1$ or -1.

A typical way of proving that a given matrix $\boldsymbol{A}$ has the unimodular property is to show that $\boldsymbol{A}$ may be constructed by a sequence of such operations from a matrix $\boldsymbol{A}'$, which in turn may be interpreted as a circulation matrix of some directed graph. Although $\boldsymbol{A}$ may itself no longer correspond directly to a directed graph, it does correspond to one of Minty's "digraphoids." Linear programming manipulations of $\boldsymbol{A}$ therefore have graphlike interpretations, which might be an important conceptual aid.

Part of our interest has been to show that many such interpretations can even be extended from unimodular subspaces to arbitrary subspaces, in terms of systems of elementary signed supports. Of course, where computational algorithms are concerned, the entirely combinatorial approach which is so efficient in graph theory must give way to a more general linear programming approach.

7. LINEAR PROGRAMMING

The results about signed supports in earlier sections of this paper place certain limitations on the patterns of signs which can occur in an equivalence class of Tucker representations. As a matter of fact, so do Tucker's results concerning linear programs. We shall apply these results now to the study of signed supports.

Tucker has shown that, starting with any tableau representing K and $K^{\perp}$, one may pass by a pivoting algorithm to a representation having one of the patterns of signs in Figure 1. In these tableaus, the top row and the leftmost column are to correspond to the same two e_i's as in the starting tableau.

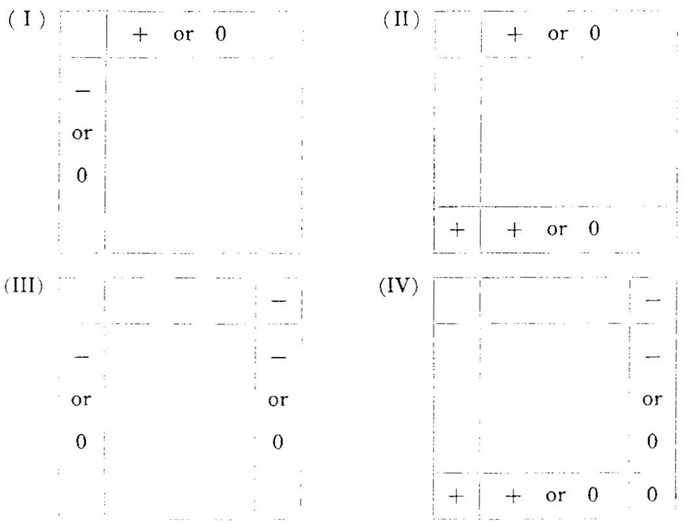

Figure 1

The four cases are mutually exclusive. In linear programming, they correspond to the cases where (I) the X problem and the Y problem have solutions, (II) the X problem is unbounded and the Y problem is inconsistent, (III) the X problem is inconsistent and the Y problem is unbounded, and (IV) the X and Y problems are both inconsistent.

The fact just described is a constructive version of the duality thorem for linear programs. But it may also be viewed, in the light of the observations of the last section, as essentially an assertion about elementary signed supports, and hence as a fundamental theorem about certain signed matroids. The bottom row in (II), for instance, corresponds to a vector in an elementary basis of K, whose support is a non-negative elementary signed support of K containing the e_i of the left-most column.

Taking (I) as alternative (a), and (II), (III) and (IV) together as (b), we can state the result as follows. Let one of the elements of E be distinguished as the "black" element and one as the "grey" element. (We have in mind the e_i of the top row in Tucker's terminal tableaus and the e_i of the left-most column, respectively.) Paint all the other elements white. Then one and only one of the following alternatives holds (and which one it is may be determined by an efficient algorithm):

(a) There exist an elementary signed support S of K containing the black element positively, and an elementary signed support S' of $K^{\perp}$ containing the grey element positively, such that no white element belongs negatively to S or to S', and no white element belongs both to S and to S'.

(b) There exists a non-negative elementary signed support of K containing the grey element but not the black element, or there exists a non-negative signed support of $K^{\perp}$ containing the black element but not the grey element, or both.

Actually, this is not quite completely contained in the result stated by Tucker, because there the black element and the grey element correspond to a row and a column initially. But that correspondence can always be arranged, unless (b) holds. For, if the grey element corresponds to a row in some Tucker representation, and that row is not entirely zeros, a simple pivoting step will calculate a new representation in which the grey element corresponds to a column. If, on the other hand, the row contains only zeros, the corresponding elementary basis of K has a vector whose support is the grey element along; this is a case of alternative (b). Similarly for the black element.

This somewhat mysterious, purely combinatorial result about signed sets, let us emphasize, has the celebrated duality theorem for linear programs as a corollary. We must therefore regard it as one of the deepest theorems possible about the signed matroids arising from subspaces of R^N. Here is an even more elaborate result, which corresponds in linear programming to the case where some constraints are equations and some variables are unconstrained.

Theorem 7. *Let one of the elements of E be painted black and one grey. Let each of the remaining elements be painted white, green or red. Then one of the following alternatives holds, but not both:*

(a) *There exists an elementary signed support S of K containing the black element positively and no red elements, and an elementary signed support S' of $K^\perp$ containing the grey element positively and no green elements, such that no white element belongs negatively to S or to S', and no white element belongs both to S and to S'.*

(b) *There exists an elementary signed support of K containing the grey element and otherwise only green or white elements, with the grey and white elements contained positively; or there exists an elementary signed support of $K^\perp$ containing the black element and otherwise only red and white elements, with the black and white elements contained positively; or both.*

Proof. This theorem must be considered known as regards linear programming, although Tucker has not discussed equality constraints or free variables explicitly in terms of his terminal tableaus. Computationally, one can decide between (a) and (b) (and construct the elementary signed supports in question) using some extension of the simplex algorithm to this more general case, such as the extension described by the writer [9]. Details will not be given here. For the sake of proving Theorem 7, however, it seems appropriate to indicate how the general case may be reduced constructively to the one previously dealt with.

Starting from an arbitrary Tucker representation, we first arrange, by simple pivoting if necessary, that the black element corresponds to a row and the grey element to a column (henceforth the "black" row and the "grey" column, etc.) (If this is not possible, then alternative (b) holds, as already explained.) We continue with simple pivoting, choosing at each step as pivot a non-zero entry in a green row and white column, or in a green row and red column, or in a white row and red column. (The consequence is that the number of red elements in D plus the number of green elements in D' in the dendroid partition of E is increased at each step.) After finitely many steps, a Tucker representation of the sort in Figure 2 is obtained (upon rearrangement of the rows and columns). The 0's mark submatrices all of whose entries are 0. (In any given example, of course, one would expect a degenerate version of this tableau, without any green rows at all, say.)

	grey	white	red	green
black			?	
white			0	
green	?	0	0	
red				

Figure 2

At this point, we look to see whether the entries in the grey column and green rows are all zero. If not, one of the green rows furnishes an elementary vector whose support satisfies alternative (b) of Theorem 7. Similarly, if the black row has a non-zero entry in a red column, then (b) holds. Otherwise we proceed with Tucker's analysis of the black-gray-white subtableau, eventually transforming it to one of the four cases in Figure 1. (At each iteration, the whole tableau is transformed in accordance with what is happening in the subtableau. The transformations trivially preserve the indicated pattern of zeros.) The conclusion, in terms of elementary signed supports, can be read from the final tableau as before.

Theorem 7 reduces to our generalization of Minty's "colored arc lemma" (Theorem 6), if one simply omits everything having to do with there being a grey element. The simplex algorithm may then be employed in practically the same way to decide constructively between the alternatives. The terminal tableaus correspond to having (I) or (III) of

Figure 1 in the upper left of Figure 2, with leftmost columns deleted. In the purely black-and-white case, as Tucker has pointed out in [15], these alternative tableaus correspond to the alternatives in the classical lemma of Farkas.

Since Theorem 7 and its algorithm are so complicated (as, indeed, they have to be to cover so many cases), a more special illstration may be helpful. Let us demonstrate how the "unsigned" form of Minty's lemma (where nothing is painted white) may be decided for an arbitrary subspace K. Here we are given a painting of E, where one element is black, and all other elements e_i are red or green. We start with any Tucker representation of K. If the black element corresponds to a column of the tableau, we look for non-zero elements in that column. If one exists, pivoting on it will yield a representation of K in which the black element corresponds to a row. If none exists, then the set consisting of the black element alone is an elementary support of K, and alternative (b) holds. Assume now that the black element corresponds to a row. We pivot next on any non-zero entry in a green row and red column. This is kept up until there are no more such pivots, at which time the tableau has the form in Figure 3.

	red	green
black	?	
green	0	
red		

Figure 3

If now the black row has a non-zero entry in some red column, the e_i's corresponding to rows with non-zero entries in that column, along with the e_i of the column itself, form an elementary support of $K^{\perp}$ containing the black element and otherwise only red elements. If, on the other hand, the black row has only 0's in red columns, then an elementary support of K is given by the black element and the green elements corresponding to columns with non-zero entries in the black row. These are alternatives (a) and (b). (Note, incidentally, that this special case of the algorithm is valid for graphoids arising from subspaces of vector spaces over *arbitrary* fields.)

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