# Monotone Operators Associated With Saddle-functions and Minimax Problems

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# MONOTONE OPERATORS ASSOCIATED WITH SADDLE-FUNCTIONS AND MINIMAX PROBLEMS

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1. **Introduction.** Let $X$ be a locally convex Hausdorff topological vector space over the real number system $R$, and let $X^*$ be the dual of $X$, with $\langle x, x^* \rangle$ written in place of $x^*(x)$ for $x \in X$ and $x^* \in X^*$. A multivalued mapping $T: X \to X^*$ is called a (nonlinear) *monotone operator* if

$$(1.1) \qquad \langle x_1 - x_2, x_1^* - x_2^* \rangle \geq 0$$

when $x_1^* \in T(x_1)$ and $x_2^* \in T(x_2)$. It is called a *maximal* monotone operator if, in addition, its graph

$$(1.2) \qquad \{(x, x^*) \mid x^* \in T(x)\} \subset X \times X^*,$$

is not contained properly in the graph of any other monotone operator $T': X \to X^*$.

One of the main classes of examples of monotone operators from $X$ to $X^*$ consists of the subdifferential mappings $\partial f$ of the proper convex functions $f$ on $X$. For $T = \partial f$, the solutions $x$ to the relation

$$(1.3) \qquad 0 \in T(x),$$

which plays a fundamental role in monotone operator theory, are the points where $f$ attains its global minimum on $X$. It is known that $\partial f$ is a maximal monotone operator when $f$ is finite and continuous throughout $X$ (Minty [**5**]), or when $X$ is a Banach space and $f$ is (proper and) lower semicontinuous throughout $X$ (Rockafellar [**12**], [**15**]).

The purpose of this paper is to present a new class of examples, the monotone operators associated with saddle-functions on $X$ (i.e. functions which are partly convex and partly concave in a sense explained below). For such a monotone operator $T$, the solutions to (1.3) are the saddle-points in a certain minimax problem. It will be proved that $T$ is maximal when the saddle-function from which it arises satisfies continuity conditions comparable to those in the case of $\partial f$.

The monotone operators associated with saddle-functions are of theoretical interest because they are closely related to extremum problems, even though they are not actually generalized gradient operators. The maximality theorems to be proved below for such operators open up a new area of applications of the theory

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of evolution equations involving monotone operators. These applications have a significance in mathematical economics.

2. **Saddle-functions.** A *convex function* on $X$ is an everywhere-defined extended-real-valued function $f$ (i.e. a function whose values are real numbers or $\pm\infty$) whose epigraph

$$\{(x, \mu) \mid x \in X, \mu \in R, \mu \geq f(x)\}, \tag{2.1}$$

is a convex set in the space $X \oplus R$. Such a function is said to be *proper* if it is not indentically $+\infty$ and if it nowhere has the value $-\infty$. A *subgradient* of a convex function $f$ at a point $x \in X$ is an $x^* \in X^*$ such that

$$f(x') \geq f(x) + \langle x' - x, x^* \rangle, \quad \forall x' \in X. \tag{2.2}$$

The (possibly empty) set of all such subgradients at $x$ is denoted by $\partial f(x)$, and the multivalued mapping $\partial f : x \to \partial f(x)$ from $X$ to $X^*$ is called the *subdifferential* of $f$.

It is known that, when $f$ is everywhere Gâteaux differentiable, $\partial f$ reduces to the (single-valued) gradient mapping $\nabla f$ from $X$ to $X^*$. More generally, if $f$ is finite and continuous at $x$, $\partial f(x)$ is a nonempty weak* compact convex subset of $X$ and

$$f'(x; x') = \max \{\langle x', x^* \rangle \mid x^* \in \partial f(x)\}, \quad \forall x' \in X, \tag{2.3}$$

where

$$\begin{aligned} f'(x; x') &= \lim_{\lambda \downarrow 0} [f(x + \lambda x') - f(x)]/\lambda \\ &= \inf_{\lambda > 0} [f(x + \lambda x') - f(x)]/\lambda. \end{aligned} \tag{2.4}$$

For proofs and further details, see Moreau [**7**], [**8**], [**9**].

An extended-real-valued function $g$ on $X$ is said to be *concave* if $-g$ is convex. It is said to be a *proper* concave function if $-g$ is a proper convex function, i.e. if $g$ is not identically $-\infty$ and $g$ nowhere has the value $+\infty$.

We assume henceforth that $X = Y \oplus Z$, where $Y$ and $Z$ are locally convex Hausdorff topological vector spaces with duals $Y^*$ and $Z^*$. We identify $X^*$ with $Y^* \oplus Z^*$ and write

$$\langle x, x^* \rangle = \langle y, y^* \rangle + \langle z, z^* \rangle$$

for $x = (y, z) \in X$ and $x^* = (y^*, z^*) \in X^*$.

By a *saddle-function* on $X$ (with respect to the given decomposition $X = Y \oplus Z$), we shall mean an everywhere-defined extended-real-valued function $K$ such that $K(y, z)$ is a concave function of $y \in Y$ for each $z \in Z$ and a convex function of $z \in Z$ for each $y \in Y$. A saddle-function $K$ will be called *proper* if there exists at least one point $x = (y, z)$ such that $K(y', z) < +\infty$ for every $y' \in Y$ and $K(y, z') > -\infty$ for every $z' \in Z$. The set of all such points will be called the *effective domain* of $K$ and denoted by dom $K$. Obviously $K$ is finite (i.e. real-valued) on dom $K$, and if $K$ is finite everywhere one has dom $K = X$.

As an example, let $L$ be any finite saddle-function on $X$, let $C$ and $D$ be non-empty convex sets in $Y$ and $Z$, respectively, and let

$$\begin{aligned} K(y, z) &= L(y, z) \quad \text{if } y \in C \text{ and } z \in D, \\ &= +\infty \quad \text{if } y \in C \text{ and } z \notin D, \\ &= -\infty \quad \text{if } y \notin C. \end{aligned} \tag{2.5}$$

Then $K$ is a proper saddle-function on $X$ with

$$\operatorname{dom} K = C \oplus D. \tag{2.6}$$

In particular, $L$ here could be any function of the form

$$L(y, z) = g(y) + h(z) + b(y, z), \tag{2.7}$$

where $g$ is a finite concave function on $Y$, $h$ is a finite convex function on $Z$ and $b$ is a bilinear function on $Y \times Z$.

The elementary but fundamental fact which motivates this paper is stated in the following theorem. This fact has previously been observed by Dantzig-Cottle [4] in the special case where $X$ is finite-dimensional and $K$ is a (finite) quadratic function (so that $T$ is a linear operator).

THEOREM 1. *Let $K$ be a proper saddle-function on $X = Y \oplus Z$, and for each $x = (y, z)$ in $X$ let $T(x) = T(y, z)$ be the set of all $x^* = (y^*, z^*)$ in $X^* = Y^* \oplus Z^*$ such that $y^*$ is a subgradient of the convex function $-K(\cdot, z)$ at $y$ and $z^*$ is a subgradient of the convex function $K(y, \cdot)$ at $z$. The multivalued mapping $T: X \to X^*$ is then a monotone operator with*

$$\{x \mid T(x) \neq \varnothing\} \subset \operatorname{dom} K. \tag{2.8}$$

PROOF. Let $(y_1^*, z_1^*) \in T(y_1, z_1)$ and $(y_2^*, z_2^*) \in T(y_2, z_2)$. By definition,

$$-K(y, z_1) \geq -K(y_1, z_1) + \langle y - y_1, y_1^* \rangle, \quad \forall y \in Y, \tag{2.9}$$

$$K(y_1, z) \geq K(y_1, z_1) + \langle z - z_1, z_1^* \rangle, \quad \forall z \in Z, \tag{2.10}$$

$$-K(y, z_2) \geq -K(y_2, z_2) + \langle y - y_2, y_2^* \rangle, \quad \forall y \in Y, \tag{2.11}$$

$$K(y_2, z) \geq K(y_2, z_2) + \langle z - z_2, z_2^* \rangle, \quad \forall z \in Z. \tag{2.12}$$

Since in particular $(y, z)$ could be a point of dom $K$, we have $-K(y_1, z_1) < +\infty$ by (2.9) and $K(y_1, z_1) < +\infty$ by (2.10). Thus $K(y_1, z_1)$ is finite, and by (2.9) and (2.10) we have $(y_1, z_1) \in \operatorname{dom} K$, establishing (2.8). By the same argument, $K(y_2, z_2)$ is finite. Taking $y = y_2$ in (2.9), $z = z_2$ in (2.10), $y = y_1$ in (2.11) and $z = z_1$ in (2.12), we get, by adding the four inequalities,

$$0 \geq \langle y_2 - y_1, y_1^* \rangle + \langle z_2 - z_1, z_1^* \rangle + \langle y_1 - y_2, y_2^* \rangle + \langle z_1 - z_2, z_2^* \rangle.$$

In other words

$$0 \leq \langle y_1 - y_2, y_1^* - y_2^* \rangle + \langle z_1 - z_2, z_1^* - z_2^* \rangle,$$

and this means that $T$ is a monotone operator.

The mapping $T$ in Theorem 1 will be called the *monotone operator associated with* $K$. (It should be noted that $T$ depends, not only on $K$ as a function on $X$, but on the given direct sum decomposition $X = Y \oplus Z$, which must be specified before the concept of "saddle-function" has meaning. There may be some other decomposition $X = Y' \oplus Z'$ with respect to which the same $K$ is a saddle-function but has a different monotone operator $T': X \to X^*$ associated with it. The decomposition $X = Y \oplus Z$ is fixed throughout the present discussion.)

The monotone operator $T$ associated with a saddle-function $K$ is closely related to the *subdifferential mapping* $\partial K$ which we have introduced elsewhere [**10**] in connection with minimax theory: namely, $\partial K$ is given in terms of $T$ by

$$\partial K(y, z) = \{(-y^*, z^*) \mid (y^*, z^*) \in T(y, z)\}. \tag{2.13}$$

For this reason, properties of $T$ have a bearing on certain extremum problems, as we shall now explain.

A point $(y, z) \in X$ is called a *saddle-point* of a saddle-function $K$ if

$$K(y', z) \leq K(y, z) \leq K(y, z'), \quad \forall y' \in Y, \forall z' \in Z, \tag{2.14}$$

i.e. if the concave function $K(\cdot, z)$ attains its maximum at $y$ and the convex function $K(y, \cdot)$ attains its minimum at $z$. It is well-known that (2.14) implies

$$K(y, z) = \sup_{y' \in Y} \inf_{z' \in Z} K(y', z') = \inf_{z' \in Z} \sup_{y' \in Y} K(y', z'). \tag{2.15}$$

In the case where $K$ is of the form (2.5), it is not hard to see that $(y, z)$ is a saddle-point of $K$ if and only if $(y, z)$ is a saddle-point of $L$ with respect to $C \times D$, i.e.

$$L(y', z) \leq L(y, z) \leq L(y, z'), \quad \forall y' \in C, \forall z' \in D, \tag{2.16}$$

in which event

$$L(y, z) = \sup_{y' \in C} \inf_{z' \in D} L(y', z') = \inf_{z' \in D} \sup_{y' \in C} L(y', z'); \tag{2.17}$$

see [**10**].

Suppose now that $T$ is the monotone operator associated with a proper saddle-function $K$. According to the definition of $T$, the relation $(y^*, z^*) \in T(y, z)$ can be expressed equivalently as

$$\begin{aligned} \langle y', y^* \rangle - \langle z, z^* \rangle + K(y', z) &\leq \langle y, y^* \rangle - \langle z, z^* \rangle + K(y, z) \\ &\leq \langle y, y^* \rangle - \langle z', z^* \rangle + K(y, z'), \quad \forall y' \in Y, \forall z' \in Z. \end{aligned} \tag{2.18}$$

But this means that $(y, z)$ is a saddle-point of the proper saddle-function $\langle \cdot, y^* \rangle - \langle \cdot, z^* \rangle + K$. In particular, the solutions $(y, z)$ to

$$(0, 0) \in T(y, z), \tag{2.19}$$

are just the saddle-points of $K$, if any. It follows that general results about the domain and range of $T$ can be interpreted as results about the existence of saddle-points, i.e. minimax theorems.

Although we shall not pursue the point here, we should like to mention that, in view of Theorem 1, the theory of monotone operators has a further bearing on minimax theory when $X$ is a Hilbert space, namely through the study of the general "evolution equation"

$$-\dot{x}(t) \in T(x(t)) \quad \text{for almost all } t, \tag{2.20}$$

where $t \to x(t)$ is (in a suitable sense) an absolutely continuous function from $[0, +\infty)$ to $X$ with derivative $t \to \dot{x}(t)$. It can be shown that, in certain cases where $X$ is finite-dimensional and $T$ is the monotone operator associated with a saddle-function $K$ of the form (2.5) with $C$ and $D$ polyhedral and $L$ differentiable, (2.20) reduces to the Arrow-Hurwicz differential equation [**1**, p. 118]. This equation and its generalizations are of interest in mathematical economics and game theory, because they describe evolution towards a state of "competitive equilibrium."

An elegant theory has already been developed concerning the existence and uniqueness of solutions to the "evolution equation" (2.20) and the convergence of such solutions to points satisfying (1.3)—see the papers of Browder and Kato in this volume. This theory requires only that $T$ be a maximal monotone operator. The maximality theorems established below will therefore make it possible to apply this theory to a new area, the study of saddle-points via generalizations of the Arrow-Hurwicz differential equation.

3. **Maximality theorems.** We shall now prove our main results, which give conditions under which the monotone operators in Theorem 1 are maximal.

THEOREM 2. *Let $K$ be a finite (i.e. everywhere real-valued) saddle-function on $X = Y \oplus Z$ such that $K(y, z)$ is everywhere separately continuous in $y$ and $z$. The monotone operator $T$ associated with $K$ is then maximal. Moreover, for each $(y, z) \in X$, $T(y, z)$ is a nonempty weak* compact convex subset of $X^*$.*

PROOF. By definition, $(y^*, z^*) \in T(y, z)$ if and only if $y^* \in C(y, z)$ and $z^* \in D(y, z)$, where $C(y, z)$ is the set of all subgradients of $-K(\cdot, z)$ at $y$ and $D(y, z)$ is the set of all subgradients of $K(y, \cdot)$ at $z$. Since the convex functions $-K(\cdot, z)$ and $K(y, \cdot)$ are finite and continuous by hypothesis, $C(y, z)$ and $D(y, z)$ are nonempty weak* compact convex subsets of $Y^*$ and $Z^*$ respectively, as indicated at the beginning of §2, and hence $T(y, z)$ is a nonempty weak* compact convex subset of $X^*$. Now fix any $(y_1, z_1) \in X$ and any $(y_1^*, z_1^*) \in X^*$ such that $(y_1^*, z_1^*) \notin T(y_1, z_1)$. We shall show that there exist a $(y_2, z_2) \in X$ and a $(y_1^*, z_1^*) \in T(y_2, z_2)$ such that

$$\langle y_2 - y_1, y_2^* - y_1^* \rangle + \langle z_2 - z_1, z_2^* - z_1^* \rangle < 0, \tag{3.1}$$

and this will establish the maximality of $T$.

Let $k$ be the real-valued function on $X \times X$ defined by

$$\begin{aligned} k(y, z; y', z') &= \max \{\langle y', y^* \rangle + \langle z', z^* \rangle \mid (y^*, z^*) \in T(y, z)\} \\ &= \max \{\langle y', y^* \rangle \mid y^* \in C(y, z)\} + \max \{\langle z', z^* \rangle \mid z^* \in D(y, z)\}. \end{aligned} \tag{3.2}$$

We note that, in view of formulas (2.3) and (2.4) (applied to the convex functions $-K(\cdot, z)$ and $K(y, \cdot)$),

$$\begin{aligned} k(y, z; y', z') &= \lim_{\lambda \downarrow 0} [-K(y + \lambda y', z) + K(y, z)]/\lambda \\ &\quad + \lim_{\lambda \downarrow 0} [K(y, z + \lambda z') - K(y, z)]/\lambda \\ &= \lim_{\lambda \downarrow 0} [K(y, z + \lambda z') - K(y + \lambda y', z)]/\lambda \\ &= \inf_{\lambda > 0} [K(y, z + \lambda z') - K(y + \lambda y', z)]/\lambda. \end{aligned} \tag{3.3}$$

Since $T(y_1, z_1)$ is a nonempty weak* closed convex set not containing $(y_1^*, z_1^*)$, we can strictly separate $(y_1^*, z_1^*)$ from $T(y_1, z_1)$ by some weak* closed hyperplane in $X^*$. Thus by (3.2) there exists some $(y', z') \in X$ such that

$$k(y_1, z_1; y', z') < \langle y', y_1^* \rangle + \langle z', z_1^* \rangle. \tag{3.4}$$

Consider the function

$$p(\theta) = k(y_1 + \theta y', z_1 + \theta z'; y', z'), \quad \theta \in R. \tag{3.5}$$

According to (3.3),

$$p(\theta) = \inf_{\lambda > 0} [K(y_1 + \theta y', z_1 + \theta z' + \lambda z') - K(y_1 + \theta y' + \lambda y', z_1 + \theta z')]. \tag{3.6}$$

Now, for any $\lambda$, the function

$$M_\lambda(\theta, \mu) = K(y_1 + \theta y', z_1 + \mu z' + \lambda z'),$$

is concave in $\theta \in R$ for each $\mu \in R$ and convex in $\mu \in R$ for each $\theta \in R$. Thus $M_\lambda$ is a finite saddle-function on $R \oplus R$. But a finite saddle-function on a finite-dimensional space is everywhere jointly continuous; this is proved in [**14**, §35]. Therefore $M_\lambda(\theta, \theta)$ is a continuous function of $\theta$. Similarly,

$$K(y_1 + \theta y' + \lambda y', z_1 + \theta z'),$$

is a continuous function of $\theta$. Formula (3.6) thus expresses $p$ as the pointwise infimum of a collection of continuous functions, and it follows that $p$ is upper semi-continuous. Hence by (3.4), since $p(0) = k(y_1, z_1; y', z')$, we must have

$$k(y_1 + \theta y', z_1 + \theta z'; y', z') < \langle y', y_1^* \rangle + \langle z', z_1^* \rangle \tag{3.7}$$

for all sufficiently small real numbers $\theta$. Fix any $\theta > 0$ for which (3.7) holds, and let

$$y_2 = y_1 + \theta y', \qquad z_2 = z_1 + \theta z'. \tag{3.8}$$

Take any $(y_2^*, z_2^*) \in T(y_2, z_2)$. The definition of $k$ implies that

$$\langle y', y_2^* \rangle + \langle z', z_2^* \rangle \le k(y_2, z_2; y', z').$$

Combining this with (3.7), we get

$$\langle y', y_2^* - y_1^*\rangle + \langle z', z_2^* - z_1^*\rangle < 0,$$

which is equivalent to (3.1) in view of (3.8). This completes the proof of Theorem 2.

COROLLARY 1. *Let $K$ be a finite saddle-function on $X = Y \oplus Z$, and suppose that $X$ is finite-dimensional. The monotone operator $T$ associated with $K$ is then maximal.*

PROOF. This is immediate from the well-known fact that a finite convex or concave function on a finite-dimensional space is necessarily continuous.

COROLLARY 2. *Let $K$ be a finite saddle-function on $X = Y \oplus Z$ which is everywhere Gâteaux differentiable, and suppose that the spaces $Y$ and $Z$ are barrelled. Express the Gâteaux gradient of $K$ by*

$$\nabla K(y, z) = (\nabla_1 K(y, z),\ \nabla_2 K(y, z)),$$

*where $\nabla_1 K(y, z) \in Y^*$ and $\nabla_2 K(y, z) \in Z^*$. The single-valued mapping*

$$(y, z) \to (-\nabla_1 K(y, z),\ \nabla_2 K(y, z)) \tag{3.9}$$

*is then a maximal monotone operator from $X$ to $X^*$.*

PROOF. The monotone operator $T$ associated with $K$ reduces to (3.9), in view of the Gâteaux differentiability of $K$. For each $y$, the convex function $K(y, \cdot)$, being Gâteaux differentiable, is the pointwise supremum of a certain collection of continuous affine functions, and hence is lower semicontinuous. But a finite lower semicontinuous convex function on a barrelled space is necessarily continuous (see [**11**]). Thus $K(y, z)$ is continuous in $z$ for each $y$. By a similar argument $K(y, z)$ is continuous in $y$ for each $z$, and it follows from the theorem that $T$ is maximal.

To get maximality results in the case of saddle-functions which are not everywhere finite, such as those of the form (2.5), more complicated continuity conditions must be imposed. These are most easily described in terms of the so-called *closure operation* for convex functions.

A convex function on $X$ is said to be *closed* if it is proper and lower semicontinuous, or else if it is one of the constant functions $+\infty$ or $-\infty$. Given any convex function $f$ on $X$, there exists a unique greatest closed convex function majorized by $f$ (the pointwise supremum of the collection of all closed convex functions majorized by $f$). This function is called the *closure* of $f$ and denoted by $\mathrm{cl}\, f$.

Given any saddle-function $K$ on $X = Y \oplus Z$, we denote by $\mathrm{cl}_2\, K$ the function on $X$ such that, for each $y \in Y$, $(\mathrm{cl}_2\, K)(y, \cdot)$ is the closure of the convex function $K(y, \cdot)$ on $Z$. Similarly, we denote by $\mathrm{cl}_1\, K$ the function on $X$ such that, for each $z \in Z$, $-(\mathrm{cl}_1\, K)(\cdot, z)$ is the closure of the convex function $-K(\cdot, z)$ on $Y$. Two saddle-functions $K$ and $K'$ are called *equivalent* if $\mathrm{cl}_1\, K = \mathrm{cl}_1\, K'$ and $\mathrm{cl}_2\, K = \mathrm{cl}_2\, K'$. A saddle-function $K$ is said to be *closed* if $\mathrm{cl}_1\, K$ and $\mathrm{cl}_2\, K$ are saddle-functions equivalent to $K$.

These notions of equivalence and closure of saddle-functions have a natural significance in minimax theory, as we have shown in [**10**], [**13**], [**14**]. For present purposes, we shall only mention a few pertinent facts. The proofs are all given in [**14**] in a finite-dimensional context, but the arguments do not actually rely on finite-dimensionality, so that they carry over immediately to arbitrary locally convex Hausdorff topological vector spaces.

The facts are as follows. Given any saddle-function $K$, the closures $\text{cl}_1 K$ and $\text{cl}_2 K$ are again saddle-functions. Furthermore, $\text{cl}_1(\text{cl}_2 K)$ and $\text{cl}_2(\text{cl}_1 K)$ are closed saddle-functions (not necessarily equivalent). If $K'$ is a saddle-function equivalent to $K$, then $\partial K' = \partial K$ (cf. (2.13)). *Thus the monotone operator $T$ associated with a proper saddle-function $K$ really depends only on the equivalence class containing $K$.* The most important fact is that the formula

$$F(y, z^*) = \sup\{\langle z, z^*\rangle - K(y, z) \mid z \in Z\}, \tag{3.10}$$

defines a one-to-one correspondence between the equivalence classes of closed proper saddle-functions $K$ on $X = Y \oplus Z$ and the lower semicontinuous proper convex functions $F$ on the space $Y \oplus Z^*$, where the topology on $Y \oplus Z^*$ is taken to be the product of the given topology on $Y$ and the Mackey topology on $Z^*$. Moreover, under this correspondence one has

$$(y^*, z^*) \in \partial K(y, z) \Leftrightarrow (-y^*, z) \in \partial F(y, z^*), \tag{3.11}$$

where $\partial F$ is the subdifferential of $F$. (Here the space of all continuous linear functionals on $Y \oplus Z^*$ in the cited topology is identified in the natural way with $Y^* \oplus Z$.) It follows that, if $K$ is a closed proper saddle-function and $T$ is the monotone operator associated with $K$, one has

$$(y^*, z^*) \in T(y, z) \Leftrightarrow (y^*, z) \in \partial F(y, z^*) \tag{3.12}$$

for the $F$ defined by (3.10). *Thus $T$ can be obtained by partial inversion of the subdifferential mapping of a certain lower semicontinuous proper convex function $F$ on $Y \oplus Z^*$.* If this subdifferential $\partial F$ is maximal, then $T$ itself must be maximal.

In particular, we get the following result.

THEOREM 3. *Let $K$ be a closed proper saddle-function on $X = Y \oplus Z$, and suppose that $Y$ and $Z$ are Banach spaces, at least one of which is reflexive. The monotone operator $T$ associated with $K$ is then maximal.*

PROOF. Suppose that $Z$ is reflexive, say. Then $Y \oplus Z^*$ is a Banach space whose dual may be identified with $Y^* \oplus Z$. Since the $F$ defined by (3.10) is a lower semicontinuous proper convex function on a Banach space, its subdifferential $\partial F$ is a maximal monotone operator (Rockafellar [**15**]). Hence, by relation (3.12), $T$ is maximal. The case where $Y$, rather than $Z$, is reflexive, can be established similarly by replacing $K$ by $-K$ and reversing the roles of the arguments $y$ and $z$.

COROLLARY 1. *Let $K$ be a proper saddle-function on $X = Y \oplus Z$ such that $K(y, z)$ is upper semicontinuous in $y$ for each $z$ and lower semicontinuous in $z$ for* *each y. Suppose that Y and Z are Banach spaces, at least one of which is reflexive. The monotone operator T associated with K is then maximal.*

PROOF. The semicontinuity conditions on $K$ imply that $K$ is closed, as is not difficult to verify using the fact that a lower semicontinuous convex function which is not proper (or an upper semicontinuous concave function which is not proper) can have no values other than $+\infty$ and $-\infty$. (Note: not every closed proper saddle-function satisfies these semicontinuity conditions, even in the case where $Y$ and $Z$ are one-dimensional; see [**14**, §34] for counterexamples.)

COROLLARY 2. *Let $K$ be a saddle-function on $X = Y \oplus Z$ of the form (2.5), where $C$ and $D$ are nonempty closed convex sets in $Y$ and $Z$, respectively, and $L$ is a finite saddle-function such that $L(y, z)$ is upper semicontinuous in $y$ for each $z$ and lower semicontinuous in $z$ for each $y$. Suppose that $Y$ and $Z$ are Banach spaces, at least one of which is reflexive. The monotone operator $T$ associated with $K$ is then maximal.*

PROOF. Here $K$ satisfies the hypothesis of Corollary 1.

4. **A counterexample.** In view of the many connections between monotone operators and convexity, it might be conjectured that, for every maximal monotone operator $T: X \to X^*$ which is not in fact the subdifferential of some convex function on $X$, there exists a direct sum decomposition $X = Y \oplus Z$ and a function $K$ on $X$ which is a saddle-function with respect to this decomposition, such that $T$ is the monotone operator associated with $K$. We shall show that this is not true even when $X$ is two-dimensional.

The counterexample we shall furnish is based on the fact that, in the finite-dimensional case at least, the set of points $(y, z)$ where $\partial K(y, z) \neq \varnothing$ is dense in dom $K$, and dom $K$ is the direct sum of a convex set in $Y$ and a convex set in $Z$ (see Rockafellar [**10**], [**14**]). This implies that the closure of the set

$$D(T) = \{(y, z) \mid T(y, z) \neq \varnothing\} \tag{4.1}$$

is the direct sum of a closed convex set in $Y$ and a closed convex set in $Z$. (Incidentally, we do not know whether $D(T)$ is dense in dom $K$ when $X$ is not finite-dimensional, although the situation in the case of purely convex functions [**2**] would suggest that this might always be true when $K$ is closed and $Y$ and $Z$ are Banach spaces.)

Let $X = R^2$, and let $A$ be the linear operator from $X$ to $X^* = R^2$ defined by

$$x = (\xi_1, \xi_2) \to x^* = (-\xi_2, \xi_1).$$

Let $B$ be the closed unit disk in $X$, and let $S$ be the subdifferential of the *indicator* of $B$, i.e. the lower semicontinuous proper convex function $f$ such that $f(x) = 0$ for $x \in B$ and $f(x) = +\infty$ for $x \notin B$. (Thus $S(x)$ consists of the zero vector alone when $x$ is an interior point of $B$, $S(x)$ consists of all the nonnegative multiples of $x$ when $x$ is a boundary point of $B$ and $S(x) = \varnothing$ when $x \notin B$.) Of course $A$ is a continuous single-valued monotone operator, whereas $S$ is a maximal monotone operator whose effective domain is $B$ [**15**]. It follows (see [**3**]) that the mapping $T: X \to X^*$ defined by $T(x) = S(x) + A(x)$ is a maximal monotone operator with $D(T) = B$. Since $D(T)$ cannot be expressed as the direct sum of two line segments, $T$ cannot arise from any saddle-function $K$, as explained above. On the other hand, $T$ is not the subdifferential of any convex function $f$ on $X$ by [**12**, Theorem 1], because $T$ reduces to $A$ on the interior of $B$ and consequently is not cyclically monotone.

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