WEAK COMPACTNESS OF LEVEL SETS OF INTEGRAL FUNCTIONALS (*)

BY

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1. Introduction

Let μ be a positive measure on a measurable space $(T, \mathcal{T})$, and let f be an extended-real-valued function on $T \times R^n$. Under rather mild regularity assumptions described below, the integral

$$I_f(u) = \int_T f(t, u(t))\mu(\mathrm{d}t) \tag{1.1}$$

is "well-defined" for every summable function $u: T \to R^n$ and thus gives an extended-real-valued functional on the Banach space $L^1(T, R^n)$. Such functionals arise in many ways, but they are especially common and important in variational problems.

In many problems one wants to minimize something of the form I_f (or a functional I such that $I \geqslant I_f$) over a subset of $L^1(T, R^n)$ defined by certain constraints. Other constraints may be represented in I_f itself by assigning the value $+\infty$ to f at "forbidden" points of $T \times R^n$, and this is how extended-real-valued functions come to be considered. Typically in the calculus of variations (or control theory) T is a region in R^m and μ is Lebesgue measure. However, there are also problems of interest where μ is an abstract probability measure (so that $I_f(u)$ is an expectation), or where μ is purely atomic (so that $I_f(u)$ is given by a series).

It is helpful in applications to know conditions under which the level sets

$$\{u \in L^1(T, R^n) \,|\, I_f(u) \leqslant \alpha\}, \quad \alpha \text{ real}, \tag{1.2}$$

are weakly compact. We shall give in Theorem 1 (§3) general conditions which are not only sufficient for this, but also, if μ is nonatomic, necessary. The sufficiency of the conditions has already been announced in a separate paper [12, Cor. 2B], but the necessity is shown here for the first time.

(*) This research was supported in part under grant no. AF-AFOSR-1202-67B at the University of Washington, Seattle.

Our proof of the compactness theorem is based on a new result (Theorem 2, §4) characterizing the weak **-closures of the level sets (1.2) as subsets of the bidual space $L^1(T, R^n)^{**}$ (the weak** topology being the weak topology induced on $L^1(T, R^n)^{**}$ by the space $L^1(T, R^n)^* = L^\infty(T, R^n)$). The argument makes much use of duality, particularly the theory of conjugate convex functions. Fundamental theorems about measurable multifunctions are also employed.

The compactness theorem is related to recent work of several authors, as we explain after its statement in §3.

2. Regularity Assumptions

The integral (1.1) does not make sense without further assumptions and conventions, since $f(t, u(t))$ might not even be measurable in t, much less summable. Classical regularity conditions are too restrictive and inflexible for the diversity of applications nowadays, but the following conditions, henceforth assumed to hold, turn out to be very natural. (For a different but closely related approach see Ioffe-Tikhomirov [4].)

(i) *The measure μ is totally σ-finite and complete.*

(ii) *f is $\mathscr{T} \times \mathscr{B}$-measurable, where $\mathscr{T} \times \mathscr{B}$ is the σ-field in $T \times R^n$ generated by the products of $\mathscr{T}$-subsets of T and Borel subsets of R^n.*

(iii) *$f(t, u)$ is lower semicontinuous as a function of $u \in R^n$ for each $t \in T$.*

Condition (ii) implies in particular that $f(t, u(t))$ is measurable in t if $u(t)$ is measurable in t. (This is evident from the measurability of the transformation $t \to (t, u(t))$.) The meaning of $I_f(u)$ for a measurable function $u: T \to R^n$ is then clear (unambiguously a real number or $+\infty$ or $-\infty$) if $f(t, u(t))$ majorizes or is majorized by a summable function of t, and in the remaining case, where neither is true, we adopt the convention that $I_f(u) = +\infty$. This convention and condition (iii) are "one-sided", but they are motivated by applications to problems of minimization.

In working with condition (ii), one is able to invoke the standard useful facts about measurable functions, but it is also possible to apply the recently developed theory of measurable multifunctions (set-valued mappings) with great advantage. A multifunction $M: T \to R^n$ (where $M(t)$ is for each $t \in T$ a subset of R^n, perhaps empty) is said to be *measurable* if the set

$$M^{-1}(C) = \{t \in T \mid M(t) \cap C \neq \emptyset\} \tag{2.1}$$

is measurable in T for every closed set $C \subset R^n$. According to an important theorem of Castaing [1] (in a somewhat extended form for R^n derived as

Theorem 1 of [13]), if $M(t)$ is closed and nonempty for every t, this condition is equivalent to the existence of a countable collection U of measurable functions $u: T \to R^n$ such that

$$M(t) = cl\{u(t) \mid u \in U\} \quad \text{for every } t \in T. \tag{2.2}$$

A theorem of Debreu [3] (again somewhat modified—see Theorem 2 of [13]) asserts on the other hand that, if $M(t)$ is closed for every t and (i) holds, the measurability of M is equivalent to the $\mathscr{T} \times \mathscr{B}$-measurability of the graph set

$$\{(t, u) \in T \times R^n \mid u \in M(t)\}.$$

The theorems of Castaing and Debreu will be used several times in this paper in connection with the following result.

LEMMA 1. *Condition* (ii) *is equivalent under* (i) *and* (iii) *to the measurability of the epigraph multifunction*

$$F: t \to F(t) = \{(u, \alpha) \in R^{n+1} \mid f(t, u) \leqslant \alpha\}. \tag{2.3}$$

PROOF. The $\mathscr{T} \times \mathscr{B}$-measurability of f is equivalent to that of the function

$$\varphi: (t, u, \alpha) \to f(t, u) - \alpha$$

and hence to that of the set

$$\{(t, u, \alpha) \in T \times R^{n+1} \mid \varphi(t, u, \alpha) \leqslant 0\},$$

which is the graph of the multifunction F. The conclusion is then apparent from Debreu's theorem, since $F(t)$ is closed for each t by condition (iii).

Throughout this paper we denote by k the greatest extended-real-valued function on $T \times R^n$ majorized by f such that $k(t, u)$ is for each t lower semicontinuous and convex in u.

(The convexity of an extended-real-valued function is defined in terms of the usual inequality by means of the obvious conventions for manipulating $+\infty$ and $-\infty$ and the special convention $+\infty - \infty = -\infty + \infty = +\infty$.)

COROLLARY. *The function k is $\mathscr{T} \times \mathscr{B}$-measurable.*

PROOF. Let K be the epigraph multifunction of k:

$$K: t \to K(t) = \{(u, \alpha) \in R^{n+1} \mid k(t, u) \leqslant \alpha\}. \tag{2.4}$$

The definition of k says that

$$K(t) = \text{cl co } F(t). \tag{2.5}$$

This and the measurability of F imply the measurability of K [13, Cor. 3.3] and hence the $\mathscr{T} \times \mathscr{B}$-measurability of k.

It follows from the corollary that, like I_f, the integral functional I_k is well-defined. The relationship between I_k and I_f will occupy much of our attention below. Note that I_k is convex by virtue of the convexity of $k(t, u)$ in u.

LEMMA 2. *If $I_f(u) < +\infty$ for at least one $u \in L^1(T, R^n)$ then*

$$\inf\{I_f(u) \mid u \in L^1(T, R^n)\} = \inf\{I_k(u) \mid u \in L^1(T, R^n)\}. \tag{2.6}$$

PROOF. The inequality $\geqslant$ certainly holds. On the other hand, suppose α is a real number such that

$$\inf\{I_k(u) \mid u \in L^1(T, R^n)\} < \alpha. \tag{2.7}$$

Then there is a summable function $\alpha_0 : T \to R^1$ such that

$$\int_T \alpha_0(t)\mu(\mathrm{d}t) < \alpha \tag{2.8}$$

and, for a certain $u_0 \in L^1(T, R^n)$,

$$k(t, u_0(t)) < \alpha_0(t) \quad \text{for almost every } t. \tag{2.9}$$

The multifunction

$$M : t \to M(t) = \{u \in R^n \mid f(t, u) \leqslant \alpha_0(t)\} \tag{2.10}$$

then has $M(t) \neq \emptyset$ for almost every t. Moreover, $M(t)$ is closed by (iii), and the graph of M is $\mathscr{T} \times \mathscr{B}$-measurable by (ii). Debreu's theorem implies that M is measurable, and from Castaing's theorem we deduce the existence of at least one measurable function $u_1 : T \to R^n$ such that

$$u_1(t) \in M(t) \quad \text{for almost every } t. \tag{2.11}$$

Since (i) holds, there is an increasing sequence of measurable sets T_m of finite measure with union T such that

$$|u_1(t)| \leqslant m \quad \text{for } t \in T_m. \tag{2.12}$$

Let u_2 be a function in $L^1(T, R^n)$ such that $I_f(u_2) < +\infty$. If m is chosen sufficiently large, the summable function

$$\begin{aligned} \beta(t) &= \alpha_0(t) \quad \text{if} \quad t \in T_m \\ &= \max\{\alpha_0(t), f(t, u_2(t))\} \quad \text{if} \quad t \notin T_m \end{aligned} \tag{2.13}$$

satisfies, in view of (2.8),

$$\int_T \beta(t)\mu(\mathrm{d}t) < \alpha. \tag{2.14}$$

Setting

$$\begin{aligned} u(t) &= u_1(t) \quad \text{if} \quad t \in T_m \\ &= u_2(t) \quad \text{if} \quad t \notin T_m \end{aligned} \tag{2.15}$$

we have $u \in L^1(T, R^n)$, and by (2.11)

$$f(t, u(t)) \leqslant \beta(t) \quad \text{for almost every } t. \tag{2.16}$$

The inequality (2.14) then implies that $I_f(u) < \alpha$. Since α was an arbitrary real number satisfying (2.7), we may conclude that equality holds in (2.6).

3. Compactness Theorem

For each $t \in T$ we denote by $g(t, \cdot)$ the (extended-real-valued) convex function on R^n conjugate to $f(t, \cdot)$ (and $k(t, \cdot)$) in the sense of Fenchel. Thus

$$\begin{aligned} g(t, v) &= \sup \{\langle u, v\rangle - f(t, u) \mid u \in R^n\} \\ &= \sup \{\langle u, v\rangle - k(t, u) \mid u \in R^n\}, \end{aligned} \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on R^n. Dually, for each t such that $k(t, u) > -\infty$ for all $u \in R^n$, we have [14]

$$k(t, u) = \sup \{\langle u, v\rangle - g(t, v) \mid v \in R^n\}. \tag{3.2}$$

The function g is $\mathscr{T} \times \mathscr{B}$-measurable; this can be seen from (3.1) by applying Castaing's theorem to the epigraph multifunction F in Lemma 1. The functional I_g is therefore well-defined and convex. Later, in our proofs, we shall treat I_g as an extended-real-valued convex functional on $L^\infty(T, R^n)$, and the duality between I_g and I_k will be important. For the present, however, we merely need to observe that $g(t, u)$ is in particular measurable in t for each u, so that the summability of $g(t, u)$ as a function of t can legitimately be considered.

We now state our main result.

THEOREM 1. *The level sets* (1.2) *of* I_f *are all weakly compact in* $L^1(T, R^n)$, *if*

(a) $g(t, v)$ *is summable in* $t \in T$ *for every* $v \in R^n$,

(b) $f(t, u)$ *is convex in* $u \in R^n$ *for almost every* $t \in T$.

Moreover, these sufficient conditions are also necessary for all the sets (1.2) *to be weakly compact, if the measure* μ *is nonatomic and* I_f *is a proper functional (that is,* $I_f(u) > -\infty$ *for every* $u \in L^1(T, R^n)$, *and* $I_f(u) < +\infty$ *for at least one* $u \in L^1(T, R^n)$).

Incidentally, if g is any real-valued function on $T \times R^n$ such that $g(t, v)$ is convex in v and condition (a) holds, then the function $f = k$ defined by (3.2) satisfies conditions (ii), (iii) and (b) ,and I_f is a proper convex functional on $L^1(T, R^n)$, this is shown in [12, Theorem 2].

As remarked in §1, we have already proved the sufficiency in Theorem 1 elsewhere [12, Corollary 2B], although a slightly different argument will be given below. An earlier proof of sufficiency [11, Theorem 4] required that $g(t, u)$ be essentially bounded in t, and that T be of finite measure. The sufficiency could also be established using the Dunford-Pettis criterion for weak compactness; the argument would be an extension of one of Valadier [16, pp. 14-16]. However, the present approach has the advantage of yielding a proof of necessity, as well as providing (in Theorem 2 below) an exact description, in the case where the level sets (1.2) are bounded in $L^1(T, R^n)$ but not weakly compact, of the weak**-compact sets which are the weak** closures of the level sets as subsets of $L^1(T, R^n)^{**}$.

If T is a bounded region in R^m and μ is Lebesgue measure, the sufficiency in Theorem 1 could also be derived from a versatile lemma of Olech [9], [10], but in a weaker form where the topology on $L^1(T, R^n)$ is that induced by certain continuous functions on T, rather than all the functions in $L^\infty(T, R^n)$. Sufficient conditions stronger in form, but valid in certain cases where R^n is replaced by an infinite-dimensional space, have been devised very recently by Castaing [2, Théorème 6].

Theorem 1 can be regarded as an extension of the classical theorem of Nagumo [8], in the sense that it furnishes (in view of the Dunford-Pettis criterion) a sufficient condition for the functions u in a set of the form (1.2) to be uniformly summable. (Condition (b) is superfluous for this conclusion.) In the classical result, T is a bounded, measurable subset of the real line and f is of the form

$$f(t, u) = \varphi(|u|), \quad u \in R^1,$$

where φ is a finite, increasing function on $[0, +\infty)$ such that

$$\lim_{\lambda \to +\infty} \varphi(\lambda)/\lambda = +\infty .$$

Our condition (a) implies that $g(t, \cdot)$ is for almost every t a finite convex function on R^n, and hence that $f(t, \cdot)$ satisfies for almost every t the growth condition

$$\lim_{\lambda \to +\infty} f(t, \lambda u)/\lambda = +\infty$$

(see [14, §13]).

Although the necessity of (a) in Theorem 1 has not previously been proved, the essential necessity of (b), even for the weak lower semicontinuity of f, is well known in various cases where f is more regular than required here. In fact, the relationship between convexity and lower semicontinuity has been an important subject of investigation in the calculus of variations ever since the discoveries of Tonelli. We cannot go into the details here, but refer the reader instead to the papers of Ioffe and Tikhomirov [4], [5], and the literature cited there.

4. Weak**-closures of Level Sets

In the method we shall use in deriving Theorem 1, the level sets (1.2) are regarded as subsets of the bidual $L^1(T, R^n)^{**}$ and compared with their closures in the weak** topology. (The restriction of the latter to $L^1(T, R^n)$, viewed as a subspace of $L^1(T, R^n)^{**}$, coincides of course with the weak topology on $L^1(T, R^n)$.)

We make use of the fact that $L^1(T, R^n)$ has a natural complement in $L^1(T, R^n)^{**}$. Specifically, $L^1(T, R^n)^{**}$ is the direct sum of the two (norm-closed) subspaces A and S, where A is the canonical copy of $L^1(T, R^n)$ and S is the set of all continuous linear functionals w on $L^\infty(T, R^n)$ with the following property: there exists an increasing sequence of measurable sets T_m of finite measure with union T, such that $w(v)=0$ whenever v is a function in $L^\infty(T, R^n)$ vanishing almost everywhere outside of T_m for some m. (This can easily be seen by representing $L^\infty(T, R^n)$ as the space of all R^n-valued continuous functions on a compact Hausdorff space $\bar{T}$ and then applying the Lebesgue decomposition theorem to the measures in the corresponding dual space.) The elements $w_A \in A$ and $w_S \in S$ in the decomposition $w = w_A + w_S$ may be termed the "absolutely continuous" and "singular" components of the functional $w \in L^1(T, R^n)^{**}$, respectively.

Using the decomposition just described (and the conventions about $+\infty$ and $-\infty$), we define the functional $\bar{I}_f$ on $L^1(T, R^n)^{**}$ by

$$\bar{I}_f(w) = I_k(w_A) + \sigma(w_S), \tag{4.1}$$

where $k(t, \cdot)$ is as in §2 the greatest lower semicontinuous, convex function on R^n majorized by $f(t, \cdot)$, and

$$\sigma(w) = \sup \{w(v) \mid v \in L^\infty(T, R^n), I_g(v) < +\infty\}. \tag{4.2}$$

Assuming that I_g is not identically $+\infty$ on $L^\infty(T, R^n)$, σ is of course a positively homogeneous, convex functional vanishing at 0. Then $\bar{I}_f$ is a convex functional on $L^1(T, R^n)^{**}$ which reduces to I_k on $L^1(T, R^n)$ (identified henceforth with A).

The following theorem, not stated before, explains the exact relationship between $\bar{I}_f$ and I_f.

THEOREM 2. *Suppose that $I_f(u) < +\infty$ for some $u \in L^1(T, R^n)$, and $I_g(v) < +\infty$ for some $v \in L^\infty(T, R^n)$. Suppose either that the measure μ is nonatomic, or that $f(t, u)$ is convex in u for almost every t. Then for every real $\alpha \neq \bar{\alpha}$, where*

$$\bar{\alpha} = \inf \{I_f(u) \mid u \in L^1(T, R^n)\}, \tag{4.3}$$

the convex level set

$$\{w \in L^1(T, R^n)^{**} \mid \bar{I}_f(w) \leq \alpha\} \tag{4.4}$$

*is the weak**-closure of the level set* (1.2). (*The set* (4.4) *is also* *weak**-closed if $\bar{\alpha}$ is real and $\alpha = \bar{\alpha}$, and in this case it is the direct sum of*

$$\{u \in L^1(T, R^n) \mid I_k(u) = \bar{\alpha}\} \tag{4.5}$$

and the convex cone consisting of the singular functionals $w \in S$ such that $w(v) \leqslant 0$ for every $v \in L^\infty(T, R^n)$ satisfying $I_g(v) < +\infty$.)

The assertion in Theorem 2 that the weak**-closure of (1.2) is (4.4) implies in particular that the weak closure of (1.2) in $L^1(T, R^n)$ is the corresponding (convex) level set of I_k. This is closely related to some results of Castaing [1] concerning measurable multifunctions, as well as to convexity theorems in the calculus of variations referred to above.

PROOF. Since I_f is not identically $+\infty$ (and likewise I_k), we have

$$\begin{aligned} I_g(v) &= \sup \{\langle u, v\rangle - I_k(u) \mid u \in L^1(T, R^n)\} \\ &= \sup \{\langle u, v\rangle - I_f(u) \mid u \in L^1(T, R^n)\} > -\infty \end{aligned} \tag{4.6}$$

for every $v \in L^\infty(T, R^n)$ by [11, Theorem 2], where

$$\langle u, v\rangle = \int_T \langle u(t), v(t)\rangle \mu(\mathrm{d}t)\,. \tag{4.7}$$

The second equality in (4.6) is seen by applying Lemma 2 to the function

$$f_v(t, u) = f(t, u) - \langle u, v(t)\rangle\,.$$

(The first equality in (4.6) is proved in [11] under the assumption, satisfied here, that I_g is not identically $+\infty$ on $L^\infty(T, R^n)$. However, we remark for purposes below that this assumption is not essential, and all of (4.6) can actually be established by nearly the same argument as used in Lemma 2.) We also have from [12, Theorem 1] the formula

$$\bar{I}_f(w) = \sup \{w(v) - I_g(v) \mid v \in L^\infty(T, R^n)\} > -\infty\,. \tag{4.8}$$

Thus $\bar{I}_f$ is the conjugate of I_g, which is not identically $+\infty$ and is the conjugate of I_f and the convex functional I_k. This implies, by the fundamental theorem about conjugate convex functions, that $\bar{I}_f$ is the greatest weak**-lower semicontinuous functional on $L^1(T, R^n)^{**}$ majorized on $L^1(T, R^n)$ by I_k, and that (since I_k is the restriction of $\bar{I}_f$ to $L^1(T, R^n)$) I_k is the greatest weakly lower semicontinuous convex functional on $L^1(T, R^n)$ majorized by I_f.

It follows immediately that

$$\begin{aligned} \inf \{\bar{I}_f(w) \mid w \in L^1(T, R^n)^{**}\} &= \inf \{I_k(u) \mid u \in L^1(T, R^n)\} \\ &= \bar{\alpha} < +\infty \end{aligned} \tag{4.9}$$

(equality with $\bar{\alpha}$ holding by Lemma 2). Furthermore, for each real $\alpha \neq \bar{\alpha}$ the level set (4.4) is the weak**-closure of the level set

$$\{u \in L^1(T, R^n) \mid I_k(u) \leqslant \alpha\}, \tag{4.10}$$

while (4.10) is in turn the weak closure in $L^1(T, R^n)$ of

$$\{u \in L^1(T, R^n) \mid (\mathrm{co}\, I_f)(u) \leqslant \alpha\}, \tag{4.11}$$

co I_f denoting the greatest convex functional majorized by I_f. If $f(t, u)$ is convex in u for almost every t, then of course co $I_f = I_f = I_k$, and we may conclude as desired that (4.4) is for $\alpha \neq \bar{\alpha}$ the weak**-closure of (1.2). The assertion of the theorem about the case $\alpha = \bar{\alpha}$ ($\bar{\alpha}$ real) needs no argument, since it is clear from (4.9) that in this case (4.4) is the set of all $w = w_A + w_S$ such that w_A belongs to (4.5) and $\sigma(w_S) \leqslant 0$.

To complete the proof of Theorem 2, we demonstrate that if $f(t, u)$ is not necessarily convex in u but μ is nonatomic, the level set (4.11) is for $\alpha \neq \bar{\alpha}$ contained in the weak closure of the corresponding level set (1.2). This amounts to demonstrating that the weak closure of the epigraph of I_f in $L^1(T, R^n) \times R^1$ is convex. Let F be the multifunction (2.3), and let

$$\tilde{F} = \{z \in L^1(T, R^{n+1}) \mid z(t) \in F(t) \text{ a.e.}\}. \tag{4.12}$$

It suffices to show that the weak closure of $\tilde{F}$ contains the convex hull co $\tilde{F}$, or equivalently, that the image of $\tilde{F}$ under an arbitrary continuous linear transformation L from $L^1(T, R^n)$ to a finite-dimensional space R^m is dense in the image of co $\tilde{F}$. Actually, it turns out that $L(\tilde{F})$ is convex, so that $L(\tilde{F}) = L(\mathrm{co}\, \tilde{F})$.

The convexity of a set of the form $L(\tilde{F})$ is a well known consequence of the theorem of Liapunov [6], according to which the range of a nonatomic R^m-valued measure is a compact convex set. (Only the convexity assertion is needed here. Lindenstrauss has furnished an elegant, half-page proof of Liapunov's theorem using the Krein-Milman theorem; the positivity hypothesis on the component measures can be obviated by means of the Hahn decomposition theorem.) The argument is standard, but we repeat it here for completeness.

Let L be a continuous linear transformation from $L^1(T, R^n)$ to R^m, and let z_1 and z_2 be elements of $\tilde{F}$. We show that $L(\tilde{F})$ includes a convex set containing $L(z_1)$ and $L(z_2)$. For each measurable subset E of T let z_E be the function in $L^1(T, R^{n+1})$ defined by

$$\begin{aligned} z_E(t) &= z_2(t) - z_1(t) \quad \text{if} \quad t \in E \\ &= 0 \quad \text{if} \quad t \notin E. \end{aligned} \tag{4.13}$$

Then $z_E + z_1 \in \tilde{F}$. The set function τ defined by

$$\tau(E) = L(z_E) \tag{4.14}$$

is countably additive from $\mathscr{T}$ to R^m (by virtue of the linearity and continuity of L), and it is also nonatomic. Hence by Liapunov's theorem the set

$$\{L(z_E) + L(z_1) \mid E \in \mathscr{T}\} \subset L(\tilde{F}) \tag{4.15}$$

is convex. This set contains $L(z_1)$ (for $E = \emptyset$) and $L(z_2)$ (for $E = T$).

5. Proof of the Compactness Theorem

Suppose in Theorem 1 that conditions (a) and (b) hold. Then $I_f = I_k$. It can be assumed that $I_f(u) < +\infty$ for some $u \in L^1(T, R^n)$, since otherwise the weak compactness of the level sets (1.2) is trivial. Then Theorem 2 is applicable. To show weak compactness it suffices to show that the sets (1.2) are bounded, and that the function σ in the definition of $\bar{I}_f$ satisfies

$$\sigma(w) = 0 \text{ for every nonzero } w \in S. \tag{5.1}$$

Given any $v \in L^\infty(T, R^n)$, we can find a finite subset $\{a_1, \ldots, a_m\}$ of R^n whose convex hull contains $v(t)$ for almost every t. Then

$$g(t, v(t)) \leqslant \max\{g(t, a_i) \mid i = 1, \ldots, m\} \tag{5.2}$$

for almost every t by virtue of the convexity of $g(t, \cdot)$. The right side of (5.2) is summable in t, and therefore $I_g(v) < +\infty$. Since v was an arbitrary element of $L^\infty(T, R^n)$, we may conclude that (5.1) holds. Furthermore, if u belongs to a level set (1.2) and $v \in L^\infty(T, R^n)$, we have

$$\langle u, v \rangle \leqslant I_f(u) + I_g(v) \leqslant \alpha + I_g(v) < +\infty \tag{5.3}$$

by (3.1), and hence the linear functional $\langle \cdot, v \rangle$ is bounded above on (1.2). Thus the sets (1.2) are all bounded, and the sufficiency of conditions (a) and (b) is proved.

Suppose now that the level sets (1.2) are all weakly compact, that I_f is "proper" as described, and that μ is nonatomic. We show that (a) and (b) must hold. Since $I_f(u) < +\infty$ for at least one $u \in L^1(T, R^n)$, formula (4.6) is valid, as already remarked in the proof of Theorem 2. Thus in particular

$$I_g(0) = -\inf\{I_f(u) \mid u \in L^1(T, R^n)\}. \tag{5.4}$$

Moreover, the infimum in (5.4) is attained, because the level sets of I_f are weakly compact, and hence the infimum cannot be $-\infty$, because $I_f(u) > -\infty$ for every $u \in L^1(T, R^n)$. Therefore $I_g(0) < +\infty$, and the hypothesis of Theorem 2 is satisfied. It follows that the sets (4.4) and (1.2) are the same for every real α.

In particular, we must have $I_f = I_k$ on $L^1(T, R^n)$. To see that this implies condition (b), consider the set $\tilde{F}$ defined in (4.12) (with F given by (2.3)), and correspondingly let

$$\tilde{K} = \{z \in L^1(T, R^{n+1}) \mid z(t) \in K(t) \text{ a.c.}\} \tag{5.5}$$

(with K given by (2.4)). Trivially $\tilde{K} \supset \tilde{F}$. Conversely, if $z \in \tilde{K}$ we have $z(t) = (u(t), \alpha(t))$, where $u \in L^1(T, R^n)$, $\alpha \in L^1(T, R^1)$, and $k(t, u(t)) \leqslant \alpha(t)$ for almost every t. Since $I_k(u) = I_f(u)$ and $k \leqslant f$, we must have

$$k(t, u(t)) = f(t, u(t)) \text{ a.e.}$$

Thus $f(t, u(t)) \leqslant \alpha(t)$ for almost every t, so that $z \in \tilde{F}$. This proves that $\tilde{K} = \tilde{F}$. We observe next that, since k is $\mathscr{T} \times \mathscr{B}$-measurable (Corollary to Lemma 1), the multifunction K is measurable (Lemma 1 applied to k). If $\bar{u}$ is a function in $L^1(T, R^n)$ such that $I_f(\bar{u}) < +\infty$, we have $k(t, \bar{u}(t))$ measurable in t and

$$k(t, \bar{u}(t)) \leqslant f(t, \bar{u}(t)) < +\infty \tag{5.6}$$

for almost every t. Thus there exists a function $\bar{z} \in \tilde{F}$ such that the set

$$T' = \{t \in T \mid \bar{z}(t) \notin K(t)\} \tag{5.7}$$

is measurable and of measure 0. Let

$$\begin{aligned} K'(t) &= K(t) \quad \text{if} \quad t \in T' \\ &= \{\bar{z}(t)\} \quad \text{if} \quad t \notin T'. \end{aligned} \tag{5.8}$$

Then $K': T \to R^{n+1}$ is a measurable multifunction whose values $K'(t)$ are closed and nonempty, and consequently there exists by Castaing's theorem a countable collection Z of measurable functions $z: T \to R^{n+1}$ such that

$$K'(t) = \text{cl}\, \{z(t) \mid z \in Z\} \text{ a.e.} \tag{5.9}$$

Since K' agrees with K except on a set of measure 0, we have

$$K(t) = \text{cl}\, \{z(t) \mid z \in Z\} \text{ a.e.} \tag{5.10}$$

In view of the fact that μ is totally σ-finite, there exists for each of the measurable functions $z \in Z$ an increasing sequence of measurable sets $T_m(z)$ of finite measure with union T, such that

$$|z(t)| \leqslant m \quad \text{for every } t \in T_m(z). \tag{5.11}$$

Let Z' denote the countable collection consisting of all the functions $z': T \to R^{n+1}$ of the form

$$\begin{aligned} z'(t) &= z(t) \quad \text{if} \quad t \in T_m(z) \\ &= \bar{z}(t) \quad \text{if} \quad t \notin T_m(z), \end{aligned} \tag{5.12}$$

where z ranges over Z, and m ranges over the positive integers. The functions in Z' are summable, and they have the property that

$$K(t) = \text{cl}\, \{z'(t) \mid z' \in Z'\} \text{ a.e.} \tag{5.13}$$

Therefore $Z' \subset \tilde{K}$. But $\tilde{K} = \tilde{F}$, as already shown, and hence (using the countability of Z') we have

$$\{z'(t) \mid z' \in Z'\} \subset F(t) \text{ a.e.} \tag{5.14}$$

The set $F(t)$ is closed by virtue of the lower semicontinuity of $f(t, u)$ in u, and therefore (5.13) and (5.14) imply

$$K(t) = F(t) \text{ a.e.} \tag{5.15}$$

This verifies condition (b).

Without loss of generality, we can assume henceforth for simplicity that $k=f$. Our task is to show that the (nonempty, convex) set

$$C=\{v\in L^\infty(T, R^n) \mid I_g(v)<+\infty\} \tag{5.16}$$

contains all the "constant functions". (For every $v\in C$ we have $I_g(v)>-\infty$ by (4.6), so that $g(t, v(t))$ is summable in t.) It is enough actually to demonstrate that the strong closure of C contains all the constant functions, for suppose the latter is true and let a be an arbitrary element of R^n. Let $\{a_1, \ldots, a_r\}$ be a subset of R^n whose convex hull is a neighborhood of a. By assumption, given any $\varepsilon>0$ we can find functions $v_i\in C$ such that

$$|v_i(t)-a_i|\leqslant \varepsilon \text{ almost everywhere } (i=1, \ldots, r). \tag{5.17}$$

If ε is sufficiently small, a will belong to the convex hull of $\{v_1(t), \ldots, v_r(t)\}$ for almost every t, so that

$$g(t, a)\leqslant \max_{i=1,\ldots,r} g(t, v_i(t)). \tag{5.18}$$

The right side of (5.18) is summable as a function of t, and therefore $a\in C$.

Assume that $\bar{v}$ is a function in $L^\infty(T, R^n)$ not belonging to the strong closure of C. We shall argue from this to a contradiction, thereby establishing Theorem 1. Since C is convex, $\bar{v}$ can be separated from C by a continuous linear functional on $L^\infty(T, R^n)$. Thus the weak**-closed, convex cone

$$D=\{w\in L^1(T, R^n)^{**} \mid w(\bar{v})\geqslant \sigma(w)\} \tag{5.19}$$

contains nonzero elements, where σ is given by (4.2). From [12, Theorem 1] we have formula (4.8), and therefore σ is the recession function ("asymptotic function") of $\bar{I}_f$ [15]. In other words, we have

$$\begin{aligned}\sigma(w) &= \lim_{\lambda\to+\infty} [\bar{I}_f(\bar{w}+\lambda w)-\bar{I}_f(\bar{w})]/\lambda. \qquad (5.20)\\ &= \sup_{\lambda>0} [\bar{I}_f(\bar{w}+\lambda w)-\bar{I}_f(\bar{w})]/\lambda,\end{aligned}$$

where $\bar{w}$ can be taken to be any element of $L^1(T, R^n)^{**}$ such that $\bar{I}_f(\bar{w})<+\infty$. In particular, (5.20) implies that

$$I_f(\bar{w}+\lambda w)\leqslant \bar{I}_f(\bar{w})+\lambda w(\bar{v}) \tag{5.21}$$

for every $w\in D$ and $\lambda>0$. Taking $\bar{w}$ to be an element of $L^1(T, R^n)$ such that $I_f(\bar{w})<\alpha$, where α is some fixed real number, and taking w to be a nonzero element of D, we see from (5.21) that $\bar{I}_f(\bar{w}+\lambda w)\leqslant\alpha$ for sufficiently small $\lambda>0$. Then $\bar{w}+\lambda w$ belongs by Theorem 2 to the weak**-closure of the level set (1.2), and hence to (1.2) itself, implying $w\in L^1(T, R^n)$. Thus D is contained in $L^1(T, R^n)$, and D consists (since

$\bar{I}_f = I_k = I_f$ on $L^1(T, R^n)$) of the functions $w \in L^1(T, R^n)$ such that

$$\int_T \langle \bar{w}(t), \bar{v}(t) \rangle \mu(dt) \geq \lim_{\lambda \to +\infty} [I_f(\bar{w} + \lambda w) - I_f(w)]/\lambda \tag{5.22}$$

$$= \lim_{\lambda \to +\infty} \int_T ([f(t, \bar{w}(t) + \lambda w(t)) - f(t, \bar{w}(t))]/\lambda) \mu(dt) .$$

(Here $\bar{w}$ still denotes an element of $L^1(T, R^n)$ such that $I_f(\bar{w}) < +\infty$.) If v is any function in C, we have

$$f(t, \bar{w}(t) + \lambda w(t)) \geq \langle \bar{w}(t) + \lambda w(t), v(t) \rangle - g(t, v(t)) \tag{5.23}$$

by (3.1), so that the difference quotient

$$[f(t, \bar{w}(t) + \lambda w(t)) - f(t, \bar{w}(t))]/\lambda, \quad \lambda > 0 , \tag{5.24}$$

majorizes a summable function of t. Furthermore, these difference quotients are monotone increasing in λ, because of the convexity of the functions $f(t, \cdot)$. It follows from the Lebesgue convergence theorem that the integral can be interchanged with the limit in (5.22). In other words, D consists of the functions $w \in L^1(T, R^n)$ such that

$$\int_T [h(t, w(t)) - \langle w(t), \bar{v}(t) \rangle] \mu(dt) \leq 0 , \tag{5.25}$$

where

$$h(t, u) = \lim_{\lambda \to +\infty} [f(t, \bar{w}(t) + \lambda u) - f(t, \bar{w}(t))]/\lambda . \tag{5.26}$$

Note that $h(t, u)$ is positively homogeneous in u.

Fix any nonzero function $w_0 \in D$. Since (5.25) holds for $w = w_0$, and since $h(t, 0) = 0$, there is a measurable set T_1 of positive measure such that

$$h(t, w_0(t)) - \langle w_0(t), \bar{v}(t) \rangle \leq 0 \text{ for almost every } t \in T_1 , \tag{5.27}$$

$$|w_0(t)| \neq 0 \text{ for almost every } t \in T_1 . \tag{5.28}$$

Let T_0 be a measurable subset of T_1 of positive, finite measure on which $|w_0(t)|$ is essentially bounded in t. Let $\mathscr{T}_0$ be the class of subsets of T_0 belonging to $\mathscr{T}$, and let μ_0 be the finite, positive, nonatomic measure on $\mathscr{T}_0$ defined by

$$\mu_0(E) = \int_E |w_0(t)| \mu(dt) . \tag{5.29}$$

Let L be the mapping which assigns to each element λ of the space $L^1(T_0, \mu_0, R^1)$ the function w given by

$$\begin{aligned} w(t) &= \lambda(t) w_0(t) \quad \text{if} \quad t \in T_0 , \\ &= 0 \quad \text{if} \quad t \notin T_0 . \end{aligned} \tag{5.30}$$

This mapping L is a linear isometry from $L^1(T_0, \mu_0, R^1)$ onto a closed subspace of $L^1(T, R^n)$. If $\lambda \geqslant 0$, the function $w = L(\lambda)$ satisfies

$$h(t, w(t)) - \langle w(t), \bar{v}(t) \rangle \leqslant 0 \text{ a.e.} \tag{5.31}$$

because of (5.27) and the positive homogeneity of $h(t, \cdot)$, and hence $w \in D$. In particular therefore, we have

$$L(B_+) \subset D, \tag{5.32}$$

where B_+ is the "nonnegative" portion of the unit ball of $L^1(T_0, \mu_0, R^1)$. Since D is weak**-closed as a subset of $L^1(T, R^n)^{**}$, the convex set $L(B_+)$, being closed and bounded, is weakly compact in $L^1(T, R^n)$. Equivalently, B_+ is weakly compact in $L^1(T_0, \mu_0, R^1)$. But this implies that $L^1(T_0, \mu_0, R^1)$ is finite-dimensional, contrary to the fact that μ_0 is positive and nonatomic.

REFERENCES

[1] Castaing, C., Sur les multi-applications mesurables, Thèse, Caen, 1967. This has partly been published in *Rev. Franc. Informatique Rech. Operationelle*, 1 (1967), 3-34.

[2] Castaing, C., *Quelques applications du théorème de Banach Dieudonné à l'Intégration*, Publication no. 67, Secrétariat des Math., Université de Montpellier.

[3] Debreu, G., Integration of correspondances, *Proc. Fifth Berkeley Symp. on Statistics and Prob.*, Vol. II, Part 1 (Univ. of California Press, Berkeley, 1966) 351-372.

[4] Ioffe, A.D. and Tikhomirov, V.M., Duality of convex functions and extremum problems, *Uspekhi Mat. Nauk*, 23 (1968), 4-116.

[5] Ioffe, A.D. and Tikhomirov, V.M., Extension of variational problems, *Trans. Moscow Math. Soc.*, 18 (1968), 207-273.

[6] Liapunov, A., On completely additive vector-functions, *Izv. Akad. Nauk S.S.S.*, R 4 (1940), 465-478.

[7] Lindenstrauss, J., A short proof of Liapounoff's convexity theorem, *J. Math. Mech.*, **15** (1966), 971-972.

[8] Nagumo, M., Über die gleichmässige Summierbarkeit und ihre Anwendung auf ein Variationsproblem, *Japan J. Math.*, *6* (1929), 178-182.

[9] Olech, C., Existence theorems for optimal problems with vector-valued cost function, *Trans. Amer. Math. Soc.*, *136* (1969), 159-180.

[10] Olech, C., Existence theorems for optimal control problems involving multiple integrals, *J. Diff. Eqs.*, *6* (1969), 512-526.

[11] Rockafellar, R.T., Integrals which are convex functionals, *Pacific J. Math.*, **24** (1968), 525-539.

[12] Rockafellar, R.T., Integrals which are convex functionals II, *Pacific J. Math*, to appear.

[13] Rockafellar, R.T., Measurable dependence of convex sets and functions on parameters, *J. Math. Anal. Appl.*, **28** (1969), 4-25.

[14] Rockafellar, R.T., *Convex Analysis*, Princeton Univ. Press, 1970.

[15] Rockafellar, R.T., Level sets and continuity of conjugate convex functions, *Trans. Amer. Math. Soc.*, *123* (1966), 46-63.

[16] Valadier, M., "Intégration des convexes fermés notamment d'épigraphes. Inf-convolution continue", Institute Rech. d'Informatique et d'Automatique, Aut. 7003.