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CONTRIBUTIONS TO NONLINEAR FUNCTIONAL ANALYSIS
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Convex Integral Functionals and Duality

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§1. Introduction.

Let $(T, \mathcal{A}, \mu)$ be a measure space, and let L be a linear space of mappings $x : T \to X$, where X is a real vector space. A convex integral functional on L is an extended-real-valued functional of the form

$$I_f(x) = \int_T f(t, x(t))\mu(dt), \qquad x \in L, \tag{1}$$

where $f(t, \cdot)$ is for each $t \in T$ an extended-real-valued convex functional on X. Such a functional is, as the name implies, convex on L, if in a rather general sense it is well-defined.

The most familiar functionals of the form I_f are undoubtedly the ones occurring in the classical theory of L^p spaces and Orlicz spaces:

$$I_f(x) = \frac{1}{p}\int_T |x(t)|^p \mu(dt), \qquad 1 \leq p < +\infty, \tag{2}$$

or more generally

$$I_f(x) = \int_T N(|x(t)|)\mu(dt), \tag{3}$$

where N is a nondecreasing convex function on $[0, +\infty)$. Here $X = R^1$. Of course, the theory of duality in L^p spaces and Orlicz spaces also involves the study of other, simpler, convex integral functionals, such as the linear functionals

$$I_f(x) = \int_T x(t)y(t)\mu(dt) \ . \tag{4}$$

These classes of functionals can be generalized by replacing $X = R^1$ by any normed linear space.

Recent work on convex integral functionals has been motivated not so much by these examples as by broader applications to the extremum problems and variational principles. For instance, many problems in the calculus of variations or optimal control involve extended-real-valued functionals of the form

$$I(z) = \int_a^b f(t, z(t), \dot{z}(t))dt \ , \tag{5}$$

where $z: [a, b] \to Z$ is a "curve" (with derivative $\dot{z}$) in a linear space Z. One can view I as a functional (1) in the case where X is $Z \times Z$ and L is the space of functions $(z, w): T \to X$ such that $w = \dot{z}$. The study of I also entails the study, for fixed choices of z, of the functional

$$w \to \int_a^b f(t, z(t), w(t))dt \ . \tag{6}$$

Even if these integral functionals are not themselves convex, it is often useful to compare them with certain convex integral functionals which they majorize, or to consider their "convexifications".

Here, instead of curves over an interval $[a, b]$, one can investigate functions z defined over a region Ω of R^m, the derivative $\dot{z}$ being replaced by a vector of partial derivatives. The generalized gradient operators associated with the functionals I (or closely related functionals) in such cases correspond to variational principles. Many important differential and integral operators, as well as other nonlinear operators on function spaces, belong to this class.

The notion of "continuous addition" of convex functions provides further motivation for a general theory of convex integral functionals. Given a collection of convex functions $f_t = f(t, \cdot)$ on X, it is possible to define another convex function

$$(7) \qquad F(x) = \int_T f(t,x)\mu(dt) ,$$

which can be regarded as the integral or "continuous sum" of the functions f_t (with respect to μ). It is an important question whether, or to what extent, certain basic results about the conjugate or subdifferential of a finite sum of convex functions can be extended to such an infinite sum. Observe that F may be identified with the restriction of I_f to the space of all constant functions $x : T \to X$.

Duality has always played a fundamental role in the analysis of convex integral functionals. One may cite in particular the classical results of Luxemburg and Zaanen [21] concerning dual functionals of the form (3). These are examples of convex functionals conjugate to each other in the sense introduced by Fenchel.

General convex integral functionals conjugate to each other were first investigated in the author's paper [26] for $X = R^n$. A concept of "normal convex integrand" was briefly developed there by elementary methods, so as to provide the necessary technical lemmas concerning measurability, measurable selections, etc. Connections between normal convex integrands and the new theory of measurable multifunctions, particularly the results of Castaing [2, 3] , Debreu [9] and Kuratowski and Ryll-Nardzewski [18], were explored in a separate paper [27]. Taking a different approach to questions of measurability, Ioffe and Tikhomirov [15, 16] studied continuous addition (7) and the dual operation of continuous infimal convolution in a separable Banach space X and its dual. The same operations were treated subsequently by Castaing [4, 5, 6] in infinite-dimensional space X and by Valadier [38] in R^n. Castaing simultaneously extended a sufficient condition of Rockafellar [26] for the weak compactness of the level sets of certain integral functionals (see also Valadier [40]). This compactness condition was also extended by the author [30] in a different manner which made it possible to show necessity, as well as sufficiency, in certain cases [31].

Among other recent work of a more special nature, not to be discussed below, we mention the comprehensive paper of Ioffe and Levin [14] on the subdifferentials of convex integral functionals such as (7). This paper is the latest in a collection including Castaing [6], Gol'shtein [11], Ioffe [12], Levin [19, 20], and Valadier [37, 39]. We mention further the papers of Ioffe [13] and Rockafellar [32, 33] dealing with applications of new results on convex integral functionals to problems in the calculus of variations. The work of Temam [35, 36] should lead to more applications of this sort.

Our aim in this paper is to set forth some of the basic theorems about convex integral functionals in a more general form than has previously appeared in the literature. For the most part, the arguments follow earlier ones, but their extension to a broader context has been made possible by the technical developments cited above, especially in the theory of measurable multifunctions.

§2. Measurable Multifunctions and Normal Integrands.

We assume henceforth that $\mathcal{A}$ is a σ-algebra of subsets of T (the measurable sets), and that μ is a positive, σ-finite measure on $\mathcal{A}$ which is complete (i.e. every subset of a set of measure zero is measurable). In this section X denotes an arbitrary complete separable metric space with metric d, and $\mathcal{B}$ denotes the σ-algebra of Borel subsets of X. The σ-algebra in $T \times X$ generated by the sets $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, is denoted by $\mathcal{A} \times \mathcal{B}$.

Given a multifunction (set-valued mapping) $\Gamma : T \to X$ and a set $S \subset X$, we denote by $\Gamma^{-1}(S)$ the set of all $t \in T$ such that $\Gamma(t) \cap S \neq \emptyset$. The set

$$D(\Gamma) = \{t \in T \mid \Gamma(t) \neq \emptyset\} = \Gamma^{-1}(X) \tag{8}$$

is called the effective domain of Γ, and the set

$$G(\Gamma) = \{(t,x) \in T \times X \mid x \in \Gamma(t)\} \tag{9}$$

the graph of Γ . We say that Γ is measurable if its graph belongs to $\mathcal{A} \times \mathcal{B}$. Other definitions of the measurability of Γ are also possible, for example in terms of the measurability of various classes of sets of the form $\Gamma^{-1}(S)$. However, in the case we are really interested in, where Γ is closed-valued, it turns out that, under our assumptions on $(T, \mathcal{A}, \mu)$ and X , all the reasonable definitions coincide.

Theorem 1. Let $\Gamma: T \to X$ be a multifunction such that $\Gamma(t)$ is a closed set for every $t \in T$. Then the following properties of Γ are equivalent.

(a) Γ is measurable, that is, $G(\Gamma)$ is a measurable set.

(b) $\Gamma^{-1}(C)$ is measurable for every closed set $C \subset X$.

(c) $\Gamma^{-1}(U)$ is measurable for every open set $U \subset X$.

(d) $\Gamma^{-1}(B)$ is measurable for every Borel set $B \subset X$.

(e) $D(\Gamma)$ is measurable, and $d(\Gamma(t), x)$ is a measurable function of $t \in D(\Gamma)$ for each $x \in X$.

(f) $D(\Gamma)$ is measurable, and there exists a countable collection $(x_i, i \in I)$ of measurable functions $x_i: D(\Gamma) \to X$ such that $\Gamma(t)$ is the closure of $\{x_i(t) \mid i \in I\}$ for each $t \in D(\Gamma)$.

(g) There exists a countable collection $(x_i, i \in I)$ of measurable functions $x_i: T \to X$ such that the set $\{x_i(t) \mid i \in I\} \cap \Gamma(t)$ is dense in $\Gamma(t)$ for each $t \in T$, and the set $\{t \in T \mid x_i(t) \in \Gamma(t)\}$ is measurable for each $i \in I$.

This theorem is the key to almost everything involving closed-valued measurable multifunctions. It is due primarily to Castaing, who states it, minus condition (g), as Lemma 2 of [7]; see also [8]. (Castaing also omits condition (b), which he used elsewhere as the definition of measurability, but the implication here from (d) to (b) to (c) is elementary.) A proof has also been furnished by Ioffe and Levin [14, Appendix II]. Most of the implications were established, at least in special cases, in Castaing's dissertation [2] and developed further in a number of papers by that author. The equivalence of (a) and (b), however, was essentially proved earlier by Debreu [9], employing arguments attributed to Freedman and Neveu. The assumptions were weakened, and various implications sharpened, for $X = R^n$ by Rockafellar [27], who introduced condition (g).

In general, the implication from (f) to (g) is trivial, while the implication from (g) to (f) can be proved by the following argument (taking the index set I to be the natural numbers). Let $T_i = \{t \in T \mid x_i(t) \in \Gamma(t)\}$. Define $x_1': D(\Gamma) \to X$ by

$$\begin{aligned} x_1'(t) &= x_1(t) \quad \text{for } t \in T_1, \\ &= x_2(t) \quad \text{for } t \in T_2 \setminus T_1, \\ &= x_3(t) \quad \text{for } t \in T_3 \setminus (T_1 \cup T_2), \text{ etc.}, \end{aligned}$$

and then for $i = 2, 3, \ldots$, define $x_i': D(\Gamma) \to X$ by

$$\begin{aligned} x_i'(t) &= x_i(t) \quad \text{for } t \in T_i, \\ &= x_1(t) \quad \text{for } t \in D(\Gamma) \setminus T_i. \end{aligned}$$

It is easily checked that the collection $(x_i', i \in I)$ has the properties in condition (f).

Some other recent work on extending Theorem 1 may be found in the dissertation of Valadier [39].

Theorem 1 is indispensible in demonstrating that measurability is achieved or preserved when multifunctions are constructed or manipulated in various ways. In most cases the ultimate purpose of all this is to enable one to invoke the following fact, obtained by specializing condition (f).

Corollary. *If* $\Gamma: T \to X$ *is a closed-valued multifunction satisfying any one of the conditions in Theorem 1, then there exists at least one measurable function* $x: T \to X$ *such that* $x(t) \in \Gamma(t)$ *for every* $t \in D(\Gamma)$.

The existence of a measurable selector x when Γ satisfies (c), a fact basic to the proof of Theorem 1, was first proved by Rokhlin in 1949 [34, Part I, §2, No. 9, Lemma 2], as Castaing has pointed out. The result was later rediscovered independently by Kuratowski and Ryll-Nardzewski [18] and Castaing [2].

It is convenient in the rest of this paper to refer to a function $f: T \times X \to (-\infty, +\infty]$ as an *integrand*. For each $t \in T$ we denote by f_t the function $t \to f(t,x)$. The *epigraph* of f_t is the set

$$\text{epi } f_t = \{(x,\alpha) \in X \times R^1 \mid f_t(x) \le \alpha\} . \tag{10}$$

Proposition 1. *The following conditions on an integrand* f *are equivalent:*

a) f *is* $\mathcal{A} \times \mathcal{B}$ *- measurable on* $T \times X$, *and for each* $t \in T$ *the function* f_t *is lower semicontinuous on* X *and not identically* $+\infty$.

b) *The multifunction* $t \to \text{epi } f_t$ *is measurable, and for each* $t \in T$ *the set* $\text{epi } f_t$ *is closed and nonempty.*

This is easily deduced, arguing by way of the measurability of the function $(t,x,\alpha) \to f(t,x) - \alpha$ and its level sets.

An integrand f satisfying the conditions in Proposition 1 is said to be *normal* . The normality property can also be expressed in terms of a condition resembling (g) of Theorem 1, and this is a useful approach in dealing with convexity; see [26, 27]. A simple criterion worth mentioning is this: f is a normal integrand if $f(t,x)$ is finite everywhere, measurable in t for fixed x, and continuous in x for fixed t. (Then the functions of the type $t \to (a, f(t,a)+\varepsilon)$ as a ranges over a countable dense subset of X and ε ranges over the positive rational numbers, form a countable collection having property (f) of Theorem 1 with respect to the multifunction $t \to \text{epi } f_t$, so that this multifunction is measurable.)

Normality ensures in particular that for every measurable function $x: T \to X$, the function $t \to f(t, x(t))$ is measurable. (The latter function is the composition of f with the measurable mapping $t \to (t, x(t)) \in T \times X$.) If the function $t \to f(t, x(t))$ is summable in the usual sense, or if it majorizes or is majorized by a summable (extended-real-valued) function on T, a natural value (possibly $+\infty$ or $-\infty$) can be assigned to the integral

(11) $$I_f(x) = \int_T f(t, x(t))\, \mu(dt).$$

In the remaining case, it has proved useful to adopt the convention that $I_f(x) = +\infty$. In this way, we regard I_f as a well-defined, extended-real-valued functional on the space of all measurable functions $x: T \to X$. The analysis of such a functional depends heavily on the effective use of the equivalences expressed in Theorem 1 and Proposition 1.

§3. The Conjugate of an Integral Functional.

We assume henceforth that X is a separable reflexive Banach space. The dual of X (which is likewise separable) is denoted by Y, and the natural bilinear pairing between elements $x \in X$ and $y \in Y$ by $\langle x, y \rangle$.

Let f be a normal integrand on $T \times X$. If f_t is convex on X for every $t \in T$, we say that f is convex. In this event I_f is a convex functional on the linear space L_0 consisting of all measurable functions $x: T \to X$.

The conjugate of the integrand f is the integrand g on $T \times Y$ defined by

(12) $$g(t, y) = \sup \{ \langle x, y \rangle - f(t, x) \mid x \in X \}.$$

According to (12), g_t is for each $t \in T$ the function on Y conjugate to f_t. The general theory of conjugate functions (see [1, 16, 22, 28]) asserts that in this case g_t is convex and lower semicontinuous. If f_t is itself convex (being already, by virtue of normality, lower semicontinuous and not identically $+\infty$), then g_t is not identically $+\infty$, and f_t is in turn the function on X conjugate to g_t:

$$f(t,x) = \sup \{\langle x, y\rangle - g(t,y) \mid y \in Y\}. \tag{13}$$

Combining such observations with a measurability argument, we obtain:

Proposition 2. The integrand g conjugate to the normal integrand f is a normal convex integrand, provided that for each t there is at least one $y \in Y$ such that $g(t,y) < +\infty$. The latter is true in particular if f is convex, and in this event f is in turn the integrand conjugate to g.

Proof. We need only show that g is measurable on $T \times Y$. To this end, we choose a countable collection of measurable multifunctions

$$t \to (x_i(t), \alpha_i(t)) \in X \times R^1, \qquad i \in I,$$

such that for each $t \in T$

$$\text{epi } f_t = c\ell \{(x_i(t), \alpha_i(t)) \mid i \in I\}.$$

Such a collection exists by property (f) of Theorem 1, since the multifunction $t \to \text{epi } f_t$ is measurable. For each $i \in I$, let

$$g_i(t,y) = \langle x_i(t), y\rangle - \alpha_i(t).$$

Then g_i is measurable on $T \times Y$, and we have

$$g(t,y) = \sup \{g_i(t,y) \mid i \in I\}.$$

The measurability of g thus follows from the fact that the pointwise supremum of a countable collection of measurable functions is measurable.

The main result of this section concerns the duality between the integral functionals I_f and I_g. Let L denote a subspace of the linear space of all measurable functions $x:T \to X$, and let M similarly denote a subspace of the linear space of all measurable functions $y: T \to Y$. We assume that $|\langle x(t), y(t)\rangle|$ is summable in t for each $x \in L$ and $y \in M$, so that the pairing

$$(14) \qquad \langle x, y\rangle_T = \int_T \langle x(t), y(t)\rangle \mu(dt), \qquad x \in L, \qquad y \in M,$$

is well-defined. As an obvious special case, one could take $L = L_X^p$ and $M = L_Y^q$ (the usual Lebesgue spaces of functions on T with values in the Banach spaces X and Y) with $1 \le p \le \infty$, $(1/p) + (1/q) = 1$.

The space L (or similarly M) is said to be <u>decomposable</u> if, whenever x belongs to L and $x_0: S \to X$ is a bounded measurable function on a measurable set $S \subset T$ of finite measure, the function

$$(15) \qquad \begin{aligned} x'(t) &= x_0(t) \text{ for } t \in S \\ &= x(t) \text{ for } t \in T/S, \end{aligned}$$

also belongs to L. The Lebesgue spaces, of course, have this property.

The following theorem, first proved by the author in [26] for $X = Y = R^n$, has not previously been stated in such generality. However, certain special infinite-dimensional cases (where X is not necessarily a separable, reflexive Banach space) have been treated by Castaing [4, 5] or are implicit in Ioffe-Tikhomirov [16] and Ioffe-Levin [14]. (Reflexivity is actually used only in the second assertion of the theorem.)

Theorem 2. *If L is decomposable and $I_f(x) < +\infty$ for at least one $x \in L$, then the convex integral functional I_g on M is conjugate to the integral functional I_f on L, that is,*

(16) $$I_g(y) = \sup\{\langle x, y\rangle_T - I_f(x) \mid x \in L\} \text{ for every } y \in M .$$

If in addition f is convex, M is decomposable and $I_g(y) < +\infty$ for at least one $y \in M$, then I_f on L is in turn conjugate to I_g on M:

(17) $$I_f(x) = \sup\{\langle x, y\rangle_T - I_g(y) \mid y \in M\} \text{ for every } x \in L .$$

Proof. It suffices to prove (16), which asserts equivalently that for each $y \in M$ the quantity

$$\int_T g(t, y(t))\,\mu(dt)$$

is the supremum of

$$\int_T [\langle x(t), y(t)\rangle - f(t, x(t))]\mu(dt)$$

over all functions $x \in L$. Replacing f_t for each t by $f_t - \langle\cdot, y(t)\rangle$ if necessary (this manipulation is normality-preserving), we can reduce the argument to the case where $y(t) \equiv 0$. Thus we need only prove that

(18) $$\inf\{\int_T f(t, x(t))\mu(dt) \mid x \in L\} = \int_T \varphi(t)\mu(dt)$$

where

(19) $$\varphi(t) = \inf\{f(t, x) \mid x \in X\} = -g(t, 0) .$$

Note that φ is a measurable function by the argument of Lemma 2. There exists by hypothesis a function $x_1 \in L$ and a summable function α_1 such that

(20) $$f(t, x_1(t)) \le \alpha_1(t) \text{ for every } t \in T .$$

Since $\varphi(t) \leq f(t, x(t))$ for every function x by definition, we see in particular that the integral of φ in (18) is in the standard sense well-defined and either finite or $-\infty$, and that the inequality $\geq$ holds in (18). Now let β be any real number such that

$$\int_T \varphi(t)\,\mu(dt) < \beta \,. \tag{21}$$

We prove the existence of a function $x \in L$ such that

$$\int_T f(t, x(t))\mu(dt) < \beta \,, \tag{22}$$

thereby establishing the theorem. From (21) (and our assumptions on the measure space) there exists a summable function α_0, such that $\varphi(t) < \alpha_0(t)$ for every t and

$$\int_T \alpha_0(t)\mu(dt) < \beta \,. \tag{23}$$

Define the multifunction $\Gamma: T \to X$ by

$$\Gamma(t) = \{x \in X \mid f(t, x) \leq \alpha_0(t)\}.$$

Since the function

$$(t, x) \to f(t, x) - \alpha_0(t)$$

is measurable, the graph of Γ is a measurable set, i.e. Γ is a measurable multifunction. Moreover, $\Gamma(t)$ is for each t closed (since f_t is lower semicontinuous) and non-empty (by (19) and the fact that $\varphi(t) < \alpha_0(t)$). The corollary of Theorem 1 then implies the existence of a measurable function x_0 (not necessarily in L) such that $x_0(t) \in \Gamma(t)$ for every t. Since (23) holds, it is possible to choose a measurable set $S \subset T$ of finite measure such that

$$\int_S \alpha_0(t)\,\mu(dt) + \int_{T \setminus S} \alpha_1(t)\mu(dt) < \beta \,. \tag{24}$$

It can be arranged at the same time that x_0 is bounded on S. Let

$$x(t) = x_0(t) \quad \text{for } t \in S,$$
$$= x_1(t) \quad \text{for } t \in T\backslash S.$$

Then $x \in L$ by the assumption of decomposability, and we have

$$f(t, x(t)) \leq \alpha_0(t) \text{ for } t \in S,$$
$$f(t, x(t)) \leq \alpha_1(t) \text{ for } t \in T \backslash S.$$

The latter implies (22), in view of (24), and the proof is complete.

Corollary. Suppose that f is convex, L and M are decomposable, and neither the functional I_f on L nor I_g on M is identically $+\infty$. Then these convex integral functionals are conjugate to each other with respect to the pairing (14), and hence in particular they are lower semi-continuous with respect to any locally convex topologies on L and M compatible with this pairing.

We remark that, in the situation in the corollary, the subdifferential mapping ∂I_f, which is a multifunction from L to M, is easily described in terms of the subdifferential mappings $\partial f_t : X \twoheadrightarrow Y$. Indeed, $\partial I_f(x)$ is for each $x \in L$ the set of all $y \in M$ such that

(25) $\qquad y(t) \in \partial f_t(x(t))$ for almost every t.

As an illustration, let us suppose that $L = L_X^p$ and $M = L_Y^q$, $1 \leq p < \infty$, $(1/p) + (1/q) = 1$. These spaces are decomposable, so that the corollary is applicable if f is any normal convex integrand such that $f(t, x(t))$ is summable in t for at least one function $x \in L_X^p$, and $g(t, y(t))$ is summable in t for at least one function $y \in L_Y^q$. The convex functionals I_f on L_X^p and I_g on L_Y^q are then lower semicontinuous with respect to not only the norm topologies, but also with respect to the weak topologies that

L^p_X and L^q_Y induce on each other. Furthermore, since L^p_X is a Banach space whose dual may be identified with L^q_Y, we can conclude that the subdifferential mapping

(26) $$\partial I_f : L^p_X \to L^q_Y \quad ,$$

which as we have seen can be expressed by the relation (25), is a maximal monotone operator, as well as a maximal cyclically monotone operator [29]. Special note should be made of the case $X = Y = R^l$, since then (25) becomes

(27) $$y(t) \in \Gamma(t, x(t)) \quad \text{for almost every } t ,$$

where $\Gamma(t, \cdot)$ is for each t a general maximal monotone operator from R^l to R^l. This case is encountered, for example, in the study of the Hammerstein equation and various boundary-value problems.

4. A Refinement, With an Application to Weak Compactness.

In the case of Theorem 1 where $L = L^1_X$ and $M = L^\infty_Y$, there is an unanswered question which turns out to be crucial in dealing with many integral functionals that arise in practice. In this case we do have, under the assumptions in the corollary above,

(28) $$I_g(y) = \sup\{\langle x, y\rangle_T - I_f(x) \mid x \in L^1_X\} \quad \text{for all } Y \in L^\infty_Y ,$$

(29) $$I_f(x) = \sup\{\langle x, y\rangle_T - I_g(y) \mid y \in L^\infty_Y\} \quad \text{for all } x \in L^1_X .$$

However, L^1_X cannot be identified with the dual $L^{\infty *}_Y$ of L^∞_Y. On $L^{\infty *}_Y$ we can define another convex functional I^* conjugate to I_g,

(30) $$I^*(y^*) = \sup\{y^*(y) - I_g(y) \mid y \in L^\infty_Y\} ,$$

and the relationship between I^* and I_f requires clarification.

Certainly if the continuous linear functional y^* is of the form

$$y^*(y) = \langle x, y\rangle_T, \qquad x \in L^1_X, \tag{31}$$

then $I^*(y^*) = I_f(x)$ by (29). Thus the restriction of I^* to the copy of L^1_X canonically embedded in $L^{\infty *}_Y$ can be identified with I_f. Put another way, I^* can be regarded as a canonical extension of the integral functional I_f to a space more general than L^1_X. What is the exact nature of the extension ? This question was answered in [30] for $X = R^n$, and the result can now also be formulated for infinite-dimensional X.

To do this, we first need to make some observations about the structure of $L^{\infty *}_Y$. A functional $y^* \in L^{\infty *}_Y$ of the type (31) is said to be <u>absolutely continuous.</u> The absolutely continuous functionals thus form a closed subspace of $L^{\infty *}_Y$ isometric to L^1_X. What is not so well known, is that this subspace has a natural complement in $L^{\infty *}_Y$, the subspace consisting of the <u>singular</u> functionals. A functional y^* is said to be singular if T can be expressed as the union of an increasing sequence of measurable sets T_m with the property that $y^*(y) = 0$ for all functions $y \in L^\infty_Y$ vanishing everywhere outside of T_m and whose essential range is totally bounded. Each $y^* \in L^{\infty *}_Y$ can be expressed uniquely as

$$y^* = y^*_a + y^*_s, \tag{32}$$

where y^*_a is absolutely continuous, y^*_s is singular and

$$\|y^*\| = \|y^*_a\| + \|y^*_s\|. \tag{33}$$

At least for $Y = R^n$, this result can be deduced by representing L^∞_Y as a space of continuous functions on a compact set and then applying the Lebesgue decomposition theorem to the elements of the dual space, regarded as measures. A more direct proof has been furnished by Dubovitskii and Miliutin [10] for $Y = R^1$, and this has been extended by Ioffe and Levin [14, Appendix I] to an arbitrary, separable Banach space Y. (Some related decomposition theorems may also be found in Ioffe [13] and Rao [24].)

Theorem 3. *Assume that* f *is convex, that* $I_f(x) < +\infty$ *for at least one* $x \in L^1_X$, *and* $I_g(y) < +\infty$ *for at least one* $y \in L^\infty_Y$. *Let* I^* *be the convex functional on* $L^{\infty *}_Y$ *defined by* (30). *Then, with respect to the canonical decomposition* (32), *one has*

$$I^*(y^*) = I_f(x) + J(y^*_s) , \tag{34}$$

where x *is the element of* L^1_X *corresponding to the absolutely continuous component* y^*_a *of* y^*. *Moreover, the functional* J *is of the special form*

$$J(y^*_s) = \sup \{y^*_s(y) \,|\, y \in D\} , \tag{35}$$

where

$$D = \{y \in L^\infty_Y \,|\, I_g(y) < +\infty\} . \tag{36}$$

Proof. The argument given by the author in [30, Theorem 1] extends ~~virtually without change~~ to the present case. (proof available on request)

A number of applications of Theorem 3, for example to integral functionals on spaces of continuous functions, have been explored in [30] for $X = Y = R^n$. We limit ourselves here to deducing from Theorem 3 a criterion for weak compactness in L^1_X.

Theorem 4. *Assume that* f *is convex, and that* $f(t, y(t))$ *is summable in* t *for every function* $y \in L^\infty_Y$. *Then for every real number* α *the convex set*

$$\{x \in L^1_X \,|\, I_f(x) \le \alpha\} \tag{37}$$

is compact in the weak topology induced on L^1_Y *by* L^∞_Y.

Proof. We can assume that $I_f(x) < +\infty$ for at least one $x \in L^1_X$, since otherwise the assertion is trivial. Then Theorem 3 is applicable, where $D = L^\infty_Y$ in (35) and (36) by our hypothesis. It follows that the set (37) can be identified with

$$\{y^* \in L^{\infty *}_Y \mid I^*(y^*) \leq \alpha^*\}, \tag{38}$$

and we need only show that the latter set is compact in the weak* topology on $L^{\infty *}_Y$. According to a basic theorem about conjugate convex functions proved at the same time by J. J. Moreau [23] and the author [25, Theorem 7A], this is true if I_g is continuous at 0 in the norm topology of L^∞_Y. Certainly I_g is lower semicontinuous throughout L^∞_Y, since I_g is conjugate to I_f (Corollary of Theorem 2). It remains only to recall that a finite, lower semicontinuous, convex function on a Banach space is necessarily continuous (see [1] or [25, Cor. 7C]).

Remark. If $X = Y = R^n$, the summability hypothesis on g in Theorem 4 can be weakened to the assumption that $g(t, y)$ is summable in t for every $y \in Y$ (or merely for every y in some dense subset of Y); see Rockafellar [30]. More generally, in the infinite-dimensional case it suffices to assume that for each real number r there is a summable function α_r such that

$$g(t, y) \leq \alpha_r(t) \quad \text{whenever} \quad y \in Y, \ \|y\| \leq r. \tag{39}$$

Theorem 4 has been proved under somewhat stronger assumptions than this by Castaing [4, 5], and the result has been sharpened further by Valadier [40]. The approach of Castaing and Valadier, based on a lemma of Grothendieck, is more direct and has the advantage of avoiding a discussion of the structure of $L^{\infty *}_Y$. On the other hand, the present approach yields additional information about the nature of the sets (38) in situations where they are not weakly compact, as well as a proof of the necessity of the condition in some cases [31].

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The preparation of this report was supported in part by grant AFOSR-71-1994.

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Received April 1, 1971